

### 1. Question

(2+2+6)= 10 points

Let  $A, B, C, D \in S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ . Two spherical triangles with vertices  $A, B, C$  and  $A, B, D$  are colunar if they share an edge and their other two edges belong to the same great circles.

- a) Triangles with vertices  $A, B, C$  and  $A, B, D$  are colunar, and  $C = (0, 0, 1)$ . Determine  $D$ .
- b) How many triangles are colunar to a given spherical triangle?
- c) Let  $\gamma$  be the parametrization of the shorter great circle arc connecting

$$A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Find an explicit formula for  $\gamma$  and show that its length satisfies

$$\cos(L(\gamma)) = \langle A, B \rangle.$$

**Overview of results:**

a)  $D = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}$ .

b) The number of triangles colunar to a given spherical triangle: .....  
*(not counting the triangle itself)* 3

c)

$$\gamma: \left[ 0, \frac{\pi}{4} \right] \rightarrow S^2, \quad \gamma(t) = \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix}$$

**Your work:**

$$L(\gamma) = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt$$

$$\gamma'(t) = \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} \Rightarrow \langle \gamma'(t), \gamma'(t) \rangle = 1$$

$$\Rightarrow L(\gamma) = \int_0^{\frac{\pi}{4}} dt = \frac{\pi}{4}$$

$$\langle A, B \rangle = \frac{1}{\sqrt{2}} = \cos(\frac{\pi}{4}).$$

## 2. Question

(4+2+4)= 10 points

Let  $H^2 = \{x \in \mathbb{R}^3 \mid \langle x, x \rangle_{2,1} = -1\}$  be the hyperboloid model of the hyperbolic plane,  $\ell_1$  and  $\ell_2$  be two hyperbolic lines and  $h_i = \{y \in H^2 \mid \langle y, n_i \rangle_{2,1} \geq 0\}, i \in \{1, 2\}$ , be halfplanes bounded by  $\ell_1$  and  $\ell_2$ . A hyperbolic rotation of  $H^2$  with centre  $x \in H^2$  is a map  $T \in SO^+(2, 1)$  with  $T(x) = x$ .

Let

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad n_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad n_2 = \begin{pmatrix} \sqrt{3} \\ -1 \\ 1 \end{pmatrix}.$$

- a) A hyperbolic rotation  $T \in SO^+(2, 1)$  with centre  $x$  maps  $h_1$  to  $h_2$ . Find  $T$ . Show your work.

Let  $\mathcal{C}$  be a Euclidean circle with radius  $0 < r < 1$  and centre at the origin in the open unit disc  $D$ . Interpreting  $D$  as the Poincaré disc model makes  $\mathcal{C}$  a hyperbolic circle.

- b) Parametrize  $\mathcal{C}$  by a curve  $\gamma$ .  
c) The Riemannian metric on the Poincaré disc model is

$$g_p(v, w) = \frac{4}{(1 - p_1^2 - p_2^2)^2} \langle v, w \rangle_{\mathbb{R}^2}, \quad p = (p_1, p_2), v, w \in D.$$

Use the Riemannian metric on  $D$  to calculate the length of  $\gamma$  as a curve in the Poincaré disc model.

**Overview of results:**

a)  $T =$

$$\begin{bmatrix} \sqrt{3}/2 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Your work:

Let  $T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ . Then

$$Tx = x \Rightarrow i = 1, c = f = 0.$$

Further,  $T \in SO(2,1) \Rightarrow T^T I_{2,1} T = I_{2,1}$

$$T^T I_{2,1} T = \begin{bmatrix} a^2 + d^2 - g^2 & ab + de - gh & -g \\ ab + de - gh & b^2 + e^2 - h^2 & -e \\ -g & -e & -1 \end{bmatrix}$$

$$\Rightarrow g = h = 0, a^2 + d^2 = 1 = b^2 + e^2, ab = -de$$

~~$$T^T I_{2,1} T = \begin{bmatrix} a^2 + d^2 & ab + de & -g \\ ab + de & b^2 + e^2 & -e \\ -g & -e & -1 \end{bmatrix}$$~~

• since  $\det T = 1$ ,  $\det \begin{bmatrix} a^2 + d^2 & ab + de & -g \\ ab + de & b^2 + e^2 & -e \\ -g & -e & -1 \end{bmatrix} = 1$

$$\Rightarrow T = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(alternatively you can use the notion of orthogonality:

$$\langle Te_1, Te_1 \rangle = a^2 + d^2 \Rightarrow \exists \alpha \text{ s.t. } (a, d) = (\cos\varphi, \sin\varphi)$$

$$\langle e_1, e_1 \rangle = 1$$

$$\langle Te_2, Te_2 \rangle = b^2 + e^2 \Rightarrow \exists \beta \text{ s.t. } (b, e) = (\sin\beta, \cos\beta)$$

$$\langle e_2, e_2 \rangle = 1$$

$$\Rightarrow 1 = \det \begin{bmatrix} a & b \\ d & e \end{bmatrix} = \cos\alpha \cos\beta - \sin\alpha \sin\beta = \cos(\alpha + \beta)$$

$$\Rightarrow \alpha = -\beta + 2\pi$$

$$\text{Let } \varphi = \alpha = -\beta + 2\pi \Rightarrow T = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know  $y \in \mathbb{R}^3 \Leftrightarrow \langle y, n_1 \rangle_{2,1} \geq 0$ , where

$$\langle \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, n_1 \rangle_{2,1} = 2y_1 - y_3$$

$y \in \mathbb{R}^3 \Leftrightarrow \langle y, n_2 \rangle_{2,1} \geq 0$ , where

$$\langle \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, n_2 \rangle_{2,1} = \sqrt{3}y_1 - y_2 - y_3$$

$$\langle T \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, n_2 \rangle_{2,1} = (\sqrt{3}\cos\varphi - \sin\varphi)y_1 - (\sqrt{3}\sin\varphi + \cos\varphi)y_2 - y_3$$

$$= \underbrace{\left\langle \begin{pmatrix} \sqrt{3}\cos\varphi - \sin\varphi \\ \sqrt{3}\sin\varphi + \cos\varphi \\ 1 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right\rangle}_{\stackrel{!}{=} n_1 \cdot \lambda}$$

e.g.  $\sqrt{3}\sin\varphi + \cos\varphi = 0$   
 $\Rightarrow \varphi = \pi - \pi/4$

$$\Rightarrow \sin\varphi = -1/2, \quad \cos\varphi = \sqrt{3}/2$$

$$b) \gamma: [0, 2\pi] \rightarrow \mathcal{C}, \quad \gamma(t) = r(\cos t, \sin t)$$

$$c) L(\gamma) = \int_0^{2\pi} g_{\gamma(t)}(\gamma'(t), \gamma'(t)) dt$$

$$\circ \gamma(t) = r(\cos t, \sin t)$$

$$\circ \gamma'(t) = r(-\sin t, \cos t)$$

$$\circ g_{\gamma(t)}(\gamma'(t), \gamma'(t)) = \frac{4}{(1 - r^2 \cos^2 t - r^2 \sin^2 t)^2} \underbrace{\langle (r \sin t), (-r \cos t) \rangle}_{= r^2}$$

$$= \frac{4r^2}{(1 - r^2)^2}$$

$$\Rightarrow L(\gamma) = \int_0^{2\pi} \frac{2r}{1 - r^2} dt = \underline{\underline{\frac{4\pi r}{1 - r^2}}}$$

### 3. Question

(3+1+3+4)= 11 points

$$q : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad q(x) = x_1^2 + 2x_1x_2 + x_1x_3 - 7x_3^2.$$

- a) Find the symmetric bilinear form  $b : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto b(x, y)$  such that  $q(x) = b(x, x)$ .
- b) What is the signature of  $q$ ?
- c) Is  $q$  non-degenerate? Show your work.
- d) Let  $e_1, e_2, e_3$  denote the canonical basis of  $\mathbb{R}^3$ . Let  $U = \text{span}\{e_1, e_1 + e_2\}$ . Find  $U^\perp$  with respect to  $b$ .

Overview of results:

a)  $b \left( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right) = x_1y_1 + x_1y_2 + y_1x_2 + \frac{1}{2}x_2y_3 + \frac{1}{2}y_1x_3 - 7x_3y_3$

b) The signature of  $q$  is  $(1, 2, 0)$

c) Is  $q$  degenerate? no

Your work: You can express  $b$  via the matrix

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -7 \end{bmatrix} \Rightarrow \det M = 7 \neq 0$$

$\Rightarrow b$  is non-degenerate.

Calculating the minors of  $M$ : let  $M_{ik} = \begin{bmatrix} m_{11} & \dots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{ik} & \dots & m_{kk} \end{bmatrix}$   
 where  $M = [m_{ij}]_{i,j=1,\dots,n}$ . Then:  
 $\det M_0 = 1$ ,  $\det M_1 = 1$ ,  $\det M_2 = -1$ ,  $\det M_3 = \det M = 7$   
 $\Rightarrow$  the signature is  $(1, 2, 0)$

$$d) U^\perp = \text{span} \left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$U^\perp = \text{span} \{ v_3 \} \quad \text{s.t. } v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{holds} \quad 0 = b(v, e_1) = b(v_1 e_1 + v_2 e_2) \\ = b(v_1 e_1) + b(v_2 e_2)$$

$$\left. \begin{array}{l} b(v_1 e_1) = v_1 + v_2 + \frac{1}{2} v_3 \\ b(v_2 e_2) = v_1 \end{array} \right\} \Rightarrow v_1 = 0, \quad v_2 = -\frac{1}{2} v_3$$

Notes and calculations:

#### 4. Question

(9)= 9 points

Decide if the following statements are true or false. Briefly justify your answer if requested.

1. (1 point) The side lengths of the polar triangle are the interior angles of the original triangle and vice versa. (Justify your work.)

True  False

They are the EXTERIOR angles.

2. (1 point) The great circles connecting  $(1, 0, 0)$  and  $(-1, 0, 0)$  in the unit sphere  $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$  correspond to lines through the origin under the stereographic projection. (Justify your work.)

True  False

Stereographic projections maps circles on  $S^2$  to circles or lines. Lines are images of those circles that pass through the north pole. ~~the~~ From the great circles in the task only the one passing through  $(1, 0, 0)$  and  $(-1, 0, 0)$  in addition will be mapped to a line.

3. (1 point) A symmetric bilinear form on a vector space  $V$  is non-degenerate if and only if the set  $\{v \in V \mid b(v, v) = 0\} = \{0\}$ . (Justify your work.)

True  False

The set  $\{v \in V \mid b(v, v) = 0\}$  does not define the kernel. A symmetric bilinear form  $b$  on  $V$  is non-degenerate if and only if  $\ker b = \{0\}$ .

Typo? should be  $\mathbb{RP}^2$ ?



4. (1 point) Let  $\mathbb{L} \subseteq \mathbb{R}^{2,1}$  be the light cone. Then  $\{[x] \in \mathbb{RP}^2 \mid x \in \mathbb{L}\}$  is a conic. (Justify your work.)

True  False

The points on  $\mathbb{L}$  satisfy

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x \in \mathbb{L} \Leftrightarrow \langle x, x \rangle_{\mathbb{R}^{2,1}} = x_1^2 + x_2^2 - x_3^2 = 0$$

thus,  $\{[x] \in \mathbb{RP}^2 \mid x \in \mathbb{L}\}$  is a conic coming from  
 $q: \mathbb{R}^3 \rightarrow \mathbb{R}, (x_1, x_2, x_3) \mapsto x_1^2 + x_2^2 - x_3^2$ .

5. (1 point) A hyperbolic line in the hyperboloid model of the hyperbolic space  $H^n \subseteq \mathbb{R}^{n,1}$  is the shortest path in the Lorentz space  $\mathbb{R}^{n,1}$  connecting two vectors  $x, y \in H^n$ . (Justify your work.)

True  False

This is VERY DIFFICULT since we did not talk about the metric on  $\mathbb{R}^{n,1}$ , but the trick is to notice that not only curves in  $H^n$  are considered.

6. (1 point) The Klein-Beltrami model of hyperbolic geometry is conformal.

True  False

7. (1 point) The projective linear group  $PGL(n, \mathbb{R})$  is a projective space. (Justify your work.)

True  False

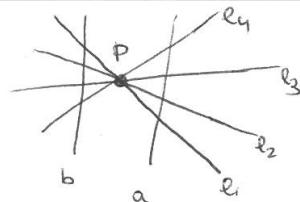
$GL(n, \mathbb{R})$  is not a vector space  
(for example, the neutral element of addition,  
 $O \in \mathbb{R}^{n \times n}$ , is not in  $GL(n, \mathbb{R})$ .)

8. (1 point) Let  $\ell_1, \dots, \ell_4$  be four lines in a projective plane  $\Pi$  intersecting in a point  $P$ . Let  $a, b \subseteq \Pi$  be two lines not containing  $P$ . Then

$$\text{cr}(a \cap \ell_1, a \cap \ell_2, a \cap \ell_3, a \cap \ell_4) = \text{cr}(b \cap \ell_1, b \cap \ell_2, b \cap \ell_3, b \cap \ell_4).$$

(Justify your work.)

a and b are related by the central projection with centre P. This projection maps  $a \cap \ell_i$  to  $b \cap \ell_i$  and is a projective transformation.  
Projective transformations preserve cross-ratios.



9. (1 point) For any five points in  $\mathbb{RP}^2$ , there is a unique conic section containing them. (Justify your work.)

This only holds if no four of the five points lie on a line.

Notes and calculations: