TECHNISCHE UNIVERSITÄT BERLIN
Faculty II - Institute of Mathematics
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Topology: Exercise Sheet 1

(Topologies, basic properties, products)

Exercise 1 (5 points)

For 2 integers a, b let $S(a, b) := \{an + b \,|\, n \in \mathbb{Z}\}$. We call a set $U \subseteq \mathbb{Z}$ open if for every $x \in U$, there is an $a \in \mathbb{Z} \setminus \{0\}$ such that $S(a, x) \subseteq U$.

(a) Show that this defines a topology on $\mathbb{Z}$. (i.e. that the collection of all open sets is a topology)

(b) Show that all sets $S(a, b)$ are closed with respect to this topology, and that $\mathbb{Z} \setminus \{-1, +1\}$ is not closed.

(c) Use these results to prove that there must be infinitely many prime numbers.

Exercise 2 (5 points)

Let X and Y be topological spaces, and let $A \subseteq X$ and $B \subseteq Y$ be equipped with subspace topology. Consider the space $X \times Y$ with product topology. Show that the product topology on $A \times B$ equals the subspace topology that $A \times B$ inherits as a subspace of $X \times Y$.

Exercise 3 (5 points)

Let S^1 be the unit sphere in $\mathbb{R}^2$ with respect to the Euclidean norm, and let $f : S^1 \to \mathbb{R}$ be continuous. Show that there is a point $x \in S^1$ such that $f(x) = f(-x)$.
(Hint: You may want to use the fact that the continuous image of a connected space is connected.)

Exercise 4 (5 points)

Let $\sim$ be the equivalence relation on $\mathbb{R}^2$ defined as follows:

$$(x, y) \sim (x', y') :\Leftrightarrow (x - x', y - y') \in \mathbb{Z}^2 \tag{1}$$

Show that $\mathbb{R}^2/\sim$ is homeomorphic to $S^1 \times S^1$.

Point total: 20