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http://www3.math.tu-berlin.de/geometrie/Lehre/WS19/CAII/ WS 19

Complex Analysis II: Riemann Surfaces

Exercise Sheet 7

(Differential forms, parallel transport, curvature)

due 13.12.2019

Exercise 1 **2 points**

Let $M = \mathbb{R}^3$. Determine which of the following forms are closed and which are exact:

(a) $\omega = yz\,dx + xz\,dy + xy\,dz\,,$

(b) $\omega = x\,dx + x^2y^2\,dy + yz\,dz\,,$

(c) $\omega = 2xy^2\,dx \wedge dy + z\,dy \wedge dz\,.$

If ω is exact, please write down the potential form θ explicitly.

Exercise 2 **2 points**

Let (E, ∇) be a smooth vector bundle with connection over a manifold M, let $\gamma\colon \mathbb{R} \to M$ be a smooth curve and let $P_t\colon E_{\gamma(0)} \to E_{\gamma(t)}$ denote the parallel transport along its restriction $\gamma|_{[0,t]}$. Show that P_t is a linear isomorphism and that

$$(\gamma^*\nabla)_{\frac{\partial}{\partial t}\big|_{t=0}} X = \tfrac{d}{dt}\big|_{t=0} P_t^{-1}\big(X(t)\big)\,,$$

for all $X \in \Gamma(\gamma^* E)$. Furthermore show that, if $\langle .,. \rangle$ is a parallel fiber metric on E ($\nabla\langle .,. \rangle = 0$), then P_t is an isometry.

Exercise 3 **2 points**

Let $M \subset \mathbb{R}^2$ be open. On $E = M \times \mathbb{R}^2$ we define two connections ∇ and $\tilde{\nabla}$ as follows:

$$\nabla = d + \begin{pmatrix} 0 & -x\,dy \\ x\,dy & 0 \end{pmatrix}, \quad \tilde{\nabla} = d + \begin{pmatrix} 0 & -x\,dx \\ x\,dx & 0 \end{pmatrix}.$$

Show that (E, ∇) is not trivial. Further construct an explicit isomorphism between $(E, \tilde{\nabla})$ and the trivial bundle (E, d).