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http://www3.math.tu-berlin.de/geometrie/Lehre/WS19/CAII/ WS 19

Complex Analysis II: Riemann Surfaces

Exercise Sheet 8

(Unitary connections, Kähler manifolds)

due 20.12.2019

Exercise 1 **2 points**

The standard hermitian metric $\langle .,. \rangle$ on the trivial bundle $\underline{\mathbb{C}}^{n+1}_{\mathbb{CP}^n}$ induces a hermitian metric on the tautological line bundle $L = \mathrm{Taut}(\mathbb{CP}^n)$. Let $\pi_L \colon \underline{\mathbb{C}}^{n+1}_{\mathbb{CP}^n} \to L$ denote the orthogonal projection in the fiber. Show that $\nabla \colon \Gamma(L) \to \Omega^1(\mathbb{CP}^n; L)$ given by

$$\nabla\psi = \pi_L(d\psi)\,,$$

defines a unitary connection on L.

Exercise 2 **2 points**

Let (M, J) be a complex manifold with hermitian Riemannian metric $\langle .,. \rangle$. The *associated Kähler form* $\omega \in \Omega^2 M$ is given by $\omega(X, Y) = \langle JX, Y \rangle$. Show that

$$d\omega(X,Y,Z) = \langle (\nabla_X J)Y, Z\rangle + \langle (\nabla_Y J)Z, X\rangle + \langle (\nabla_Z)X, Y\rangle\,,$$
$$2\langle (\nabla_X J)Y, Z\rangle = d\omega(X,Y,Z) - d\omega(X, JY, JZ)\,.$$

Conclude that M is Kähler if and only if $d\omega = 0$.

Hint: On a complex manifold X, Y, Z, JY and JZ can be assumed to be commuting vector fields.

Exercise 3 **2 points**

Show that a complex submanifold of a Kähler manifold is Kähler.