# Differential Geometry II
## Analysis and geometry on manifolds
## Exercise Sheet 10

*Due:* **January 16, 2020**

**[Points Total: 20]**

**Exercise 1: [3+3 points]**

The Lie derivative $L_X Y$ is one notion of derivatives for vector fields.

a) Show that the map $L : (X, Y) \mapsto L_X Y$ is not a connection on $TM$.

b) Show that there are vector fields $V, W$ on $\mathbb{R}^2$ such that $V = W = \partial_1$ along the $x^1$-axis but the Lie derivatives $L_V \partial_2$ and $L_W \partial_2$ are not equal on the $x^1$-axis. (This shows that Lie differentiation does not provide a well-defined way to take derivatives of vector fields along curves.)

**Exercise 2: [5 points]**

Let $M$ be a smooth manifold and $(N, g)$ be a Riemannian manifold with Levi-Civita connection $\nabla$. If $f : M \to N$ is a diffeomorphism, we define the pullback $f^*\nabla$ of $\nabla$ by

$$(f^*\nabla)_X Y := f_*^{-1}(\nabla_{f_* X} f_* Y)$$

for all $X, Y \in \Gamma(TM)$. Show that $f^*\nabla$ is the Levi-Civita connection on $M$ for the pullback metric $f^*g$.

**Exercise 3: [2+2 points]**

Let $S \subset \mathbb{R}^3$ be the surface given by $z = xy$, parametrized by the coordinates $x, y$. [That is, $S = \{(x, y, z) \in \mathbb{R}^3 | z = xy\}$ and $(\mathbb{R}^2, \phi)$ is a coordinate chart for $S$ where $\phi : \mathbb{R}^2 \to S$, $\phi(x, y) = (x, y, xy)$.]

a) Find the the induced Riemannian metric on $S$ in coordinates $x, y$.

b) Compute the Christoffel symbols for the Levi-Civita connection on $S$.

**Exercise 4: [3+2 points]**

Let $M$ be a smooth manifold and $\nabla$ be a connection on $TM$. For a curve $\gamma$ in $M$ from $p \in M$ to $q \in M$, the mapping $P_\gamma : T_pM \to T_qM$ defined by $P_\gamma(X_p) = X_q$, where $X_q$ is the unique extension of $X_p$ to a parallel vector field along $\gamma$, is called the parallel transport along $\gamma$.

a) Show that the parallel transport is an invertible linear map.

b) Show that if $M$ is an oriented Riemannian manifold with Levi-Civita connection $\nabla$, then the parallel transport is an orientation-preserving isometry.