Exercise 1: Midcircles. (3 pts)
Show that for any pair of circles $C_1$ and $C_2$ in the plane there is a circle $C$ such that inversion in $C$ maps $C_1$ to $C_2$ and vice versa. Is it unique?

Exercise 2: Steiner porism. (2 pts)
A Steiner chain is a finite sequence of circles $C_1, \ldots, C_n$, where each circle is tangent to two given non-intersecting circles $A, B$, and every circle in the chain is tangent to the previous and next circle in the chain. If the first and $n$-th circle are also tangent, the chain is called closed. Prove the following theorem, called Steiner Porism:

If there exists at least one closed Steiner chain of $n$ circles for two given non-intersecting circles $A$ and $B$, then there is an infinite number of closed Steiner chains of $n$ circles; and any circle tangent to $A$ and $B$ in the same way is a member of such a chain.

Exercise 3: Fixed points of Möbius transformations. (3 pts)
Note: In the following the word Möbius transformation always refers to orientation preserving Möbius transformations.

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation, i.e.,

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Prove the following statements:

(i) If $f$ has exactly one fixed point, then there exists a Möbius transformation $g$ and $b \in \mathbb{C} \setminus \{0\}$ such that

$$(g \circ f \circ g^{-1})(z) = z + b.$$ 

(ii) If $f$ has exactly two distinct fixed points, then there exists a Möbius transformation $g$ and $a \in \mathbb{C} \setminus \{0, 1\}$ such that

$$(g \circ f \circ g^{-1})(z) = az.$$