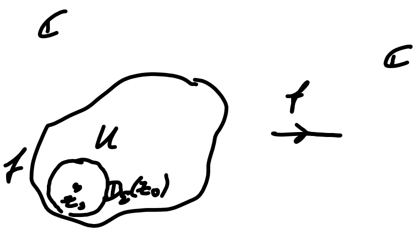
Holomorphic functions and residues



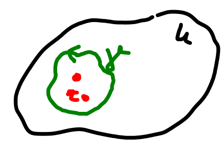
- $f: \mathbb{C} \supset U \longrightarrow \mathbb{C}$ holomorphic $\iff$ f complex differentiable
- locally ($z_0 \in U$, $D_\varepsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\} \subset U$) a holomorphic function f has a series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k .$$

- for $z_0 \notin U$ an isolated singularity of f ($D_\varepsilon^*(z_0) = \{z \in \mathbb{C} / 0 < |z - z_0| < \varepsilon\} \subset U$) a holomorphic function f has a series expansion (Laurent series)

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

- z_0 is a pole of order m $\iff$ $a_{-m} \neq 0$ and $a_k = 0$ for $k < -m$
- the residue of f at z_0 is given by



$$\operatorname*{res}_{z=z_0} f(z) = a_{-1} = \frac{1}{2\pi i} \int_\gamma f(z)\,dz$$

- for a 1^{st} order pole z_0 the residue is equal to

$$\operatorname*{res}_{z=z_0} f(z) = \lim_{z \to z_0} \big((z - z_0) f(z) \big) \qquad (*)$$

$$f(z) = \sum_{k=-1}^{\infty} a_k (z - z_0)^k = \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots$$

$$(z - z_0) f(z) = a_{-1} + a_0 (z - z_0) + a_1 (z - z_0)^2 + a_2 (z - z_0)^3 + \ldots$$

$$\lim_{z \to z_0} \big((z - z_0) f(z) \big) = a_{-1}$$

In the lecture we had

$$g(\lambda) = \frac{x^2}{a + \lambda} + \frac{y^2}{b + \lambda} - 1$$

g has two 1^{st} order poles at $\lambda = -a$ and $\lambda = -b$

$$\operatorname*{res}_{\lambda = -a} g(\lambda) = x^2 \quad , \quad \operatorname*{res}_{\lambda = -b} g(\lambda) = y^2$$

We now consider the 2^{nd} degree polynomial

$$p(\lambda) := g(\lambda)(a + \lambda)(b + \lambda) = x^2 (b + \lambda) + y^2 (a + \lambda) - (a + \lambda)(b + \lambda) = -\lambda^2 + \ldots$$

We know $g(u_1) = g(u_2) = 0$, and thus, the roots of p are u_1 and u_2

$$p(\lambda) = -(\lambda - u_1)(\lambda - u_2)$$

This yields the following form for g:

$$g(\lambda) = \frac{-(\lambda - u_1)(\lambda - u_2)}{(a+\lambda)(b+\lambda)}$$

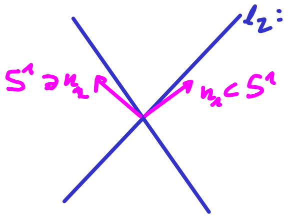
We now compute the residues of g using (*)

$$\operatorname*{res}_{\lambda=-a} g(\lambda) = \lim_{\lambda \to -a} \left((\lambda + a)\, g(\lambda) \right) = \lim_{\lambda \to -a} \left(\frac{-(\lambda-u_1)(\lambda-u_2)}{(b+\lambda)} \right) = \frac{-(-a-u_1)(-a-u_2)}{b-a} = \frac{(a+u_1)(a+u_2)}{a-b}$$

$$\operatorname*{res}_{\lambda=-b} g(\lambda) = \frac{-(-b-u_1)(-b-u_2)}{a-b} = \frac{(b+u_1)(b+u_2)}{b-a}.$$

Circle points and orthogonality

$\mathbb{R}^2 \subset \mathbb{R}P^2$



$\in S^1 \subset \mathbb{R}^2$ Euclidean scalar product $\mathbb{R}$

$\ell_2 : \langle n_1, \binom{x}{y} \rangle = d_1$

$\iff n_1^1 x + n_1^2 y = d$

$\iff n_1^1 x_1 + n_1^2 x_2 - d x_3 = 0$

$S^1 \ni n_2$ $\quad n_1 \in S^1$

$\ell_1 : n_2^1 x_1 + n_2^2 x_2 - d_2 x_3 = 0$

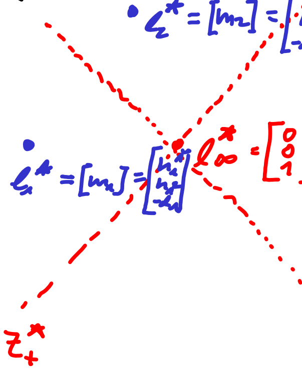
Z_+ $\quad Z_-$ line at infinity ℓ_∞ $\quad x_3 = 0$

Z: circle points $\begin{cases} x_3 = 0 \\ x_1^2 + x_2^2 = 0 \end{cases}$

$$Z_\pm = \begin{bmatrix} 1 \\ \pm i \\ 0 \end{bmatrix}$$

$\overset{\text{dualize}}{\longleftrightarrow}$

$(\mathbb{R}P^2)^*$



$\bullet\ \ell_2^* = [m_2] = \begin{bmatrix} n_2^1 \\ n_2^2 \\ -d_2 \end{bmatrix} = \begin{bmatrix} n_1 \\ -d \end{bmatrix}$

$\bullet\ \ell_1^* = [m_1] = \begin{bmatrix} n_1^1 \\ n_1^2 \\ -d_1 \end{bmatrix}$ $\quad \ell_\infty^* = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Z_+^* $\quad Z_-^*$

Z^*: dual circle points: $\tilde{x}_1^2 + \tilde{x}_2^2 = 0$

(the solution to this equation are exactly the two lines Z_+^*: $x_1 + i x_2 = 0$

Z_-^*: $x_1 - i x_2 = 0$

the corresp. symmetric bilinear form is

$q(\tilde{x}, \tilde{y}) = \tilde{x}_1 \tilde{y}_1 + \tilde{x}_2 \tilde{y}_2$

Prop.: $\quad \ell_1 \perp \ell_2 \quad \iff \quad \ell_1^*, \ell_2^*$ conjugate w.r.t. the conic Z^*

$\iff \quad q(m_1, m_2) = 0$

$\quad \parallel$

$q\left(\binom{n_1}{-d_1}, \binom{n_2}{-d_2} \right) = \langle n_1, n_2 \rangle$