Theorem 1.12:

Let $\phi: \mathbb{R}^2 \supset I_1 \times I_2 \longrightarrow \mathbb{R}^2$, $\phi(s_1,s_2) = \begin{pmatrix} x(s_1,s_2) \\ y(s_1,s_2) \end{pmatrix}$ be a coordinate system.

ϕ confocal coordinate system:

$\exists u_1: I_1 \to \mathcal{I}_1 = (-a,-b)$, $u_2: I_2 \to \mathcal{I}_2 = (-b,\infty)$

s.t. (*) $\begin{cases} \dfrac{x(s_1,s_2)^2}{a+u_1(s_1)} + \dfrac{y(s_1,s_2)^2}{b+u_1(s_1)} = 1 \\[2ex] \dfrac{x(s_1,s_2)^2}{a+u_2(s_2)} + \dfrac{y(s_1,s_2)^2}{b+u_2(s_2)} = 1 \end{cases}$

$\Longleftrightarrow$

$\exists f_1, g_1 : I_1 \to \mathbb{R}$, $f_2, g_2 : I_2 \to \mathbb{R}$

with $\begin{cases} f_1^2 + g_1^2 = a-b \\ f_2^2 - g_2^2 = a-b \end{cases}$

and $\begin{cases} x(s_1,s_2) = \dfrac{f_1(s_1) f_2(s_2)}{\sqrt{a-b}} \\[2ex] y(s_1,s_2) = \dfrac{g_1(s_1) g_2(s_2)}{\sqrt{a-b}} \end{cases}$

Proof:

"$\Rightarrow$" $f_1(s_1) := \sqrt{a+u_1(s_1)}$, $f_2(s_2) := \sqrt{a+u_2(s_2)}$

$g_1(s_1) := \sqrt{-(b+u_1(s_1))}$ $g_2(s_2) := \sqrt{b+u_2(s_2)}$

Then $f_1^2 + g_1^2 = a+u_1 - (b+u_1) = a-b$

$f_2^2 - g_2^2 = a+u_2 - (b+u_2) = a-b$

and $x(s_1,s_2) = \dfrac{\sqrt{a+u_1(s_1)}\sqrt{a+u_2(s_2)}}{\sqrt{a-b}}$

$y(s_1,s_2) = \dfrac{\sqrt{-(b+u_1(s_1))}\sqrt{b+u_2(s_2)}}{\sqrt{a-b}}$

$\Rightarrow$ $x(s_1,s_2)^2 = \dfrac{(a+u_1(s_1))(a+u_2(s_2))}{a-b}$

$y(s_1,s_2)^2 = \dfrac{(b+u_1(s_1))(b+u_2(s_2))}{b-a}$ $\Leftrightarrow$ (*)

"$\Leftarrow$" $u_1(s_1) := f_1(s_1)^2 - a$, $u_2(s_2) := f_2(s_2)^2 - a$

Then $u_1(s_1) = a-b-g_1(s_1)^2 - a = -(b+g_1(s_1)^2)$

$u_2(s_2) = a-b+g_2(s_2)^2 - a = g_2(s_2)^2 - b$

Then $x(s_1,s_2)^2 = \dfrac{(a+u_1(s_1))(a+u_2(s_2))}{a-b}$

$y(s_1,s_2)^2 = \dfrac{(b+u_1(s_1))(b+u_2(s_2))}{b-a}$ $\Leftrightarrow$ (*) .

Theorem 1.13:

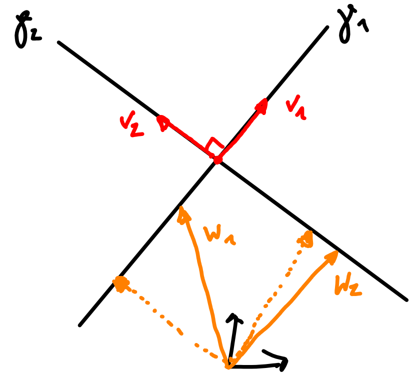
Φ factorizes and is orthogonal $\Rightarrow$ Φ confocal coordinate system

Proof: (last part)

We arrived at

$$\frac{x(s_1,s_2)^2}{\tilde{a}_1(s_1)} + \frac{y(s_1,s_2)^2}{\tilde{b}_1(s_1)} = 1$$

$$\frac{x(s_1,s_2)^2}{\tilde{a}_2(s_2)} + \frac{y(s_1,s_2)^2}{\tilde{b}_2(s_2)} = 1$$



$$\tilde{a}_1(s_1) - \tilde{b}_1(s_1) = \frac{\langle v_1, w_2\rangle \det(w_1, v_1)}{v_1^1 v_1^2}$$

$$\tilde{\alpha}_2(s_2) - \tilde{b}_2(s_2) = \frac{\langle v_2, w_1\rangle \det(w_2, v_2)}{v_2^1 v_2^2}$$

$\gamma_1(s_1) = \alpha_1(s_1) v_1 + w_1$

$\gamma_2(s_1) = \alpha_2(s_2) v_2 + w_2$

$\langle v_1, v_2\rangle = 0$

We have to show $\tilde{a}_1 - \tilde{b}_1 = \tilde{a}_2 - \tilde{b}_2$.

We can make different choices for v_1, v_2, w_1, v_2 by e.g.

$\tilde{\alpha}_1(s_1) := \frac{1}{\lambda}\alpha_1(s_1)$ with some $\lambda \neq 0$ $\rightsquigarrow$ $\gamma_1(s_1) = \tilde{\alpha}_1(s_1) \underbrace{\lambda v_1}_{=:\tilde{v}_1} + w_1$

$\tilde{\alpha}_1(s_1) := \alpha_1(s_1) - c$ with some $c \in \mathbb{R}$ $\rightsquigarrow$ $\gamma_1(s_1) = \tilde{\alpha}_1(s_1) v_1 + \underbrace{c v_1 + w_1}_{=:\tilde{w}_1}$

choose c s.t.: $\langle c v_1 + w_1, v_1\rangle = 0 \iff c\underbrace{\langle v_1, v_1\rangle}_{=1} + \langle w_1, v_1\rangle = 0$

$\iff c = -\langle w_1, v_1\rangle$

Using this we choose

$\|v_1\| = \|v_2\| = 1$, $\det(v_1, v_2) = 1$, $w_1 = \lambda_1 v_2$, $w_2 = \lambda_2 v_1$ for some $\lambda_1, \lambda_2 \neq 0$

Then

$$(1) = \frac{\langle v_1, w_2\rangle \det(w_1, v_1)}{v_1^1 v_1^2} = \frac{\lambda_2 \overbrace{\|v_1\|^2}^{=1} \lambda_1 \overbrace{\det(v_2, v_1)}^{=-1}}{v_1^1 v_1^2} = -\frac{\lambda_1\lambda_2}{v_1^1 v_1^2}$$

$$(2) = \frac{\langle v_2, w_1\rangle \det(w_2, v_2)}{v_2^1 v_2^2} = \frac{\lambda_1 \overbrace{\|v_2\|^2}^{=1} \lambda_2 \overbrace{\det(v_1, v_2)}^{=1}}{v_2^1 v_2^2} = \frac{\lambda_1\lambda_2}{v_2^1 v_2^2}$$

$$(1) = (2) \iff \frac{-1}{v_1^1 v_1^2} = \frac{1}{v_2^1 v_2^2} \iff v_1^1 v_1^2 + v_2^1 v_2^2 = 0$$

$$\begin{pmatrix} v_1^1 & v_2^1 \\ v_1^2 & v_2^2 \end{pmatrix} \in O(2)$$

$$\iff \begin{pmatrix} v_1^1 & v_1^2 \\ v_2^1 & v_2^2 \end{pmatrix} \in O(2)$$

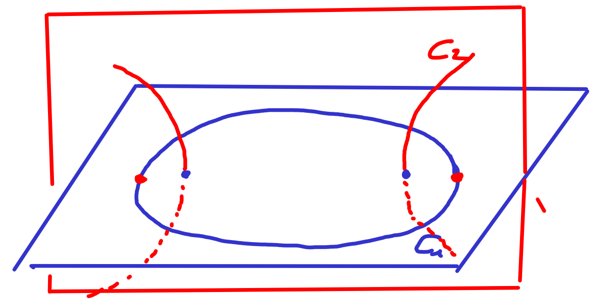
$$\Rightarrow \left\langle \begin{pmatrix} v_1^1 \\ v_2^1 \end{pmatrix}, \begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix} \right\rangle = v_1^1 v_1^2 + v_2^1 v_2^2 = 0$$

We still need to check that $\tilde{a}_1 - \tilde{b}_1 \neq 0$.

Assume $\tilde{a}_1 = \tilde{b}_1$. Since $\langle v_1, w_2\rangle \neq 0$, this means $\det(w_1, v_1) = 0$ ↯

Focal conics

Theorem 2.1:



$$C_1: \frac{x^2}{a} + \frac{y^2}{b} = 1 \quad , \ a > b$$

$$C_2: \frac{x^2}{\tilde{a}} + \frac{z^2}{\tilde{b}} = 1 \quad , \ \tilde{a} > \tilde{b}$$

Then C_1 and C_2 are focal conics if and only if

$$\tilde{a} = a - b, \quad \tilde{b} = -b.$$

Proof: The foci of C_1 are given by $(\pm\sqrt{a-b}, 0, 0) = F_\pm$

$$F_\pm \in C_2 \iff \frac{a-b}{\tilde{a}} + 0 = 1 \iff \tilde{a} = a - b$$

The foci of C_2 are given by $(\pm\sqrt{\tilde{a}-\tilde{b}}, 0, 0) = G_\pm$

$$G_\pm \in C_1 \iff \frac{\tilde{a}-\tilde{b}}{a} + 0 = 1 \iff \tilde{b} = \tilde{a} - a = -b$$