Confocal coordinates using Jacobi elliptic functions

1) Solve $\quad f_1^2(s_1) + g_1^2(s_1) = a-b \quad , \quad f_1^2(s_1) + h_1^2(s_1) = a-c$

using $\quad cn^2 + sn^2 = 1 \quad , \quad dn^2 + k^2 sn^2 = 1 \quad , \quad 0 < k^2 < 1$

Ansatz:
$$f_1(s_1) = \alpha_1 sn(s_1, k_1)$$
$$g_1(s_1) = \beta_1 cn(s_1, k_1)$$
$$h_1(s_1) = \gamma_1 dn(s_1, k_1)$$

We obtain: $\quad \alpha_1^2 sn^2 + \beta_1^2 cn^2 = a-b \quad , \quad \alpha_1^2 sn^2 + \gamma_1^2 dn^2 = a-c$

Determine $\alpha_1, \beta_1, \gamma_1, k_1$ such that

$$\frac{\alpha_1^2}{a-b} = 1 \quad , \quad \frac{\beta_1^2}{a-b} = 1 \qquad \frac{\alpha_1^2}{a-c} = k_1^2 \quad , \quad \frac{\gamma_1^2}{a-c} = 1$$

$\hookrightarrow \quad \alpha_1 = \sqrt{a-b} \quad , \quad \beta_1 = \sqrt{a-b} \quad , \quad k_1 = \frac{\sqrt{a-b}}{\sqrt{a-c}} \quad , \quad \gamma_1 = \sqrt{a-c} = \frac{\sqrt{a-b}}{k_1}$

$(0 < k_1 < 1)$

2) Solve $\quad f_2^2(s_2) - g_2^2(s_2) = a-b \quad , \quad f_2^2(s_2) + h_2^2(s_2) = a-c$

using $\quad dn^2 + k^2 sn^2 = 1 \quad , \quad dn^2 + k^2(1-cn^2) = 1$

$$\Longleftrightarrow \quad dn^2 - k^2 cn^2 = 1 - k^2$$

Ansatz:
$$f_2(s_2) = \alpha_2 dn(s_2, k_2)$$
$$g_2(s_2) = \beta_2 cn(s_2, k_2)$$
$$h_2(s_2) = \gamma_2 sn(s_2, k_2)$$

We obtain: $\quad \alpha_2^2 dn^2 - \beta_2^2 cn^2 = a-b \quad , \quad \alpha_2^2 dn^2 + \gamma_2^2 sn^2 = a-c$

Determine $\alpha_2, \beta_2, \gamma_2, k_2$ such that

$$\frac{\alpha_2^2}{a-b} = \frac{1}{1-k_2^2} \quad , \quad \frac{\beta_2^2}{a-b} = \frac{k_2^2}{1-k_2^2} \quad , \quad \frac{\alpha_2^2}{a-c} = 1 \quad , \quad \frac{\gamma_2^2}{a-c} = k_2^2$$

$\hookrightarrow \quad \alpha_2^2 = a-c \quad , \quad k_2^2 = 1 - \frac{a-b}{\alpha_2^2} = 1 - \frac{a-b}{a-c} = 1 - k_1^2 = \frac{b-c}{a-c} \quad (0 < k_2 < 1)$

$$\beta_2^2 = (a-b)\frac{k_2^2}{1-k_2^2} = (a-b)\,\frac{b-c}{a-c}\,\frac{a-c}{a-b} = b-c$$

$$\gamma_2^2 = (a-c)\,k_2^2 = a-c\,\frac{b-c}{a-c} = b-c$$

$$\alpha_2 = \sqrt{a-c} = \frac{\sqrt{b-c}}{k_2} \quad , \quad \beta_2 = \sqrt{b-c} \quad , \quad \gamma_2 = \sqrt{b-c} \quad , \quad k_2 = \sqrt{1-k_1^2} = \frac{\sqrt{b-c}}{\sqrt{a-c}}$$

3) Solve $\quad f_3^2(s_3) - g_3^2(s_3)^2 = a-b \quad , \quad f_3^2(s_3) - h_3^2(s_3) = a-c$

using $\quad cn^2 + sn^2 = 1 \quad , \quad dn^2 + k^2 sn^2 = 1 \quad \Big/ sn^2$

$$\Leftrightarrow \frac{cn^2}{sn^2} + 1 = \frac{1}{sn^2} \quad , \quad \frac{dn^2}{sn^2} + k^2 = \frac{1}{sn^2}$$

$$\Rightarrow \frac{1}{sn^2} - \frac{cn^2}{sn^2} = 1 \quad , \quad \frac{1}{sn^2} - \frac{dn^2}{sn^2} = k^2$$

$$\Leftrightarrow ns^2 - cs^2 = 1 \quad , \quad ns^2 - ds^2 = k^2$$

Ansatz: $f_3(s_3) = \alpha_3 ns(s_3, k_3)$

$g_3(s_3) = \beta_3 ds(s_3, k_3)$

$h_3(s_3) = \gamma_3 cs(s_3, k_3)$

We obtain: $\alpha_3^2 ns^2 - \beta_3^2 ds^2 = a-b \quad , \quad \alpha_3^2 ns^2 - \gamma_3^2 cs^2 = a-c$

Determine $\alpha_3, \beta_3, \gamma_3, k_3$ s.t.

$$\frac{\alpha_3^2}{a-b} = \frac{1}{k_3^2} \quad , \quad \frac{\beta_3^2}{a-b} = \frac{1}{k_3^2} \quad , \quad \frac{\alpha_3^2}{a-c} = 1 \quad , \quad \frac{\gamma_3^2}{a-c} = 1$$

$$\alpha_3^2 = \gamma_3^2 = a-c$$

$$k_3^2 = \frac{a-b}{\alpha_3^2} = \frac{a-b}{a-c} = k_1^2$$

$$\beta_3^2 = \frac{a-b}{k_3^2} = a-c$$

$$\alpha_3 = \beta_3 = \gamma_3 = \sqrt{a-c} \quad , \quad k_3 = k_1 = \frac{\sqrt{a-b}}{\sqrt{a-c}}$$