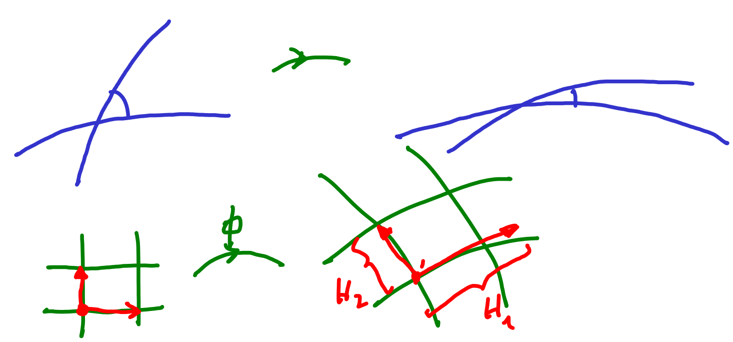
Lamé coefficients of confocal coordinates (in 2D)

$$x(u_1,u_2) = \frac{\sqrt{a+u_1}\sqrt{a+u_2}}{\sqrt{a-b}}, \qquad y(u_1,u_2) = \frac{\sqrt{-(b+u_1)}\sqrt{b+u_2}}{\sqrt{a-b}} \qquad (1)$$

$$\partial_1 x = \frac{\sqrt{a+u_2}}{2\sqrt{a+u_1}\sqrt{a-b}}, \qquad \partial_1 y = -\frac{\sqrt{b+u_2}}{2\sqrt{-(b+u_1)}\sqrt{a-b}}$$

$$\begin{aligned}
H_1^2 = \left\|\begin{pmatrix}\partial_1 x\\ \partial_1 y\end{pmatrix}\right\|^2 &= (\partial_1 x)^2 + (\partial_1 y)^2 \\
&= \frac{1}{4(a-b)}\left(\frac{a+u_2}{a+u_1} - \frac{b+u_2}{b+u_1}\right) \\
&= \frac{1}{4(a-b)} \frac{(a+u_2)(b+u_1) - (a+u_1)(b+u_2)}{(a+u_1)(b+u_1)} \\
&= \frac{1}{4(a-b)} \frac{au_1 + bu_2 - au_2 - bu_1}{(a+u_1)(b+u_1)} \\
&= \frac{1}{4(a-b)} \frac{(a-b)(u_1-u_2)}{(a+u_1)(b+u_1)} \\
&= \frac{(u_1-u_2)}{4(a+u_1)(b+u_1)}
\end{aligned}$$



Similarly,

$$H_2^2 = \frac{u_2-u_1}{4(a+u_2)(b+u_2)}.$$

In particular,

$$(*) \qquad \frac{H_1^2(u_1,u_2)}{H_2^2(u_1,u_2)} = \frac{(a+u_2)(b+u_2)}{(a+u_1)(b+u_1)} = \frac{\alpha_1(u_1)}{\alpha_2(u_2)} \quad \text{with} \quad \alpha_i(u_i) = \frac{(-1)^i}{(a+u_i)(b+u_i)}, \; i=1,2$$

How do the Lamé coefficients change under reparametrization along coordinate lines?

$$\tilde{\Phi}(s_1,s_2) = \phi(u_1(s_1), u_2(s_2))$$

$$\begin{aligned}
\tilde{H}_1^2(s_1,s_2) &= \left\|\frac{\partial}{\partial s_1}\tilde{\Phi}\right\|^2 = \left\|\frac{\partial}{\partial s_1}\phi(u_1(s_1),u_2(s_2))\right\|^2 = \left\|u_1'(s_1)\frac{\partial}{\partial u_1}\phi(u_1(s_1),u_2(s_2))\right\|^2 \\
&= u_1'(s_1)^2 H_1^2(u_1(s_1),u_2(s_2))
\end{aligned}$$

For (*) this means:

$$\frac{\hat{H}_1^2(s_1,s_2)}{\hat{H}_2^2(s_1,s_2)} = \frac{u_1'(s_1)^2\, H_1^2}{u_2'(s_1)^2\, H_2^2} = \frac{u_1'(s_1)^2\, \alpha_1(u_1(s_1))}{u_2'(s_1)^2\, \alpha_2(u_2(s_2))}$$

If we choose u_1, u_2 s.t.

$$u_1'(s_1)^2 = \frac{1}{\alpha(u_1(s_1))} \quad \text{and} \quad u_2'(s_2)^2 = \frac{1}{\alpha(u_2(s_2))}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$u_1'^2 = (a+u_1)(b+u_1) \qquad u_2'^2 = -(a+u_2)(b+u_2) \qquad (**)$$

Let us also compute the Lamé coefficients for

$$x(s_1,s_2) = \sqrt{a-b}\,\cos s_1 \cosh s_2 \quad , \quad y(s_1,s_2) = \sqrt{a-b}\,\sin s_1 \sinh s_2$$

$$H_1^2(s_1,s_2) = (\partial_1 x)^2 + (\partial_1 y)^2 = (a-b)\left(\sin^2 s_1 \cosh^2 s_2 + \cos^2 s_1 \sinh^2 s_2\right)$$

$$H_2^2(s_1,s_2) = (\partial_2 x)^2 + (\partial_2 y)^2 = (a-b)\left(\cos^2 s_1 \sinh^2 s_2 + \sin^2 s_1 \cosh^2 s_2\right)$$

In particular, $H_1^2 = H_2^2$ (conformal parametrization)

To obtain the parametrization from (1) we choose:

$$u_1(s_1) = f_1(s_1)^2 - a = (a-b)\cos^2 s_1 - a$$

$$u_2(s_2) = f_2(s_2)^2 - a = (a-b)\cosh^2 s_2 - a$$

which are solutions to $(**)$.

Euler-Poisson-Darboux system for confocal coordinates

$$x^2 = \frac{(a+u_1)(a+u_2)(a+u_3)}{(a-b)(a-c)} \quad , \quad y^2 = \frac{(b+u_1)(b+u_2)(b+u_3)}{(b-a)(b-c)} \quad , \quad z^2 = \frac{(c+u_1)(c+u_2)(c+u_3)}{(c-a)(c-b)}$$

From the first equation we obtain (by taking the part. derivative w.r.t. u_1):

$$2x\,\partial_1 x = \frac{(a+u_2)(a+u_3)}{(a-b)(a-c)} = \frac{x^2}{a+u_1} \quad \Longleftrightarrow \quad \partial_1 x = \frac{x}{2(a+u_1)}$$

Similarly, $\partial_i x = \frac{x}{2(a+u_i)}$, $\partial_i y = \frac{y}{2(b+u_i)}$, $\partial_i z = \frac{z}{2(c+u_i)}$,

and thus f $i \neq j$

$$\partial_i \partial_j x = \frac{\partial_j x}{2(a+u_i)} = \frac{x}{4(a+u_i)(a+u_j)}$$

On the other hand

$$\frac{1}{2(u_i-u_j)}(\partial_j x - \partial_i x) = \frac{1}{2(u_i-u_j)}\left(\frac{x}{2(a+u_j)} - \frac{x}{2(a+u_i)}\right)$$

$$= \frac{x}{4(u_i-u_j)}\underbrace{\left(\frac{1}{a+u_j} - \frac{1}{a+u_i}\right)}_{= \frac{a+u_i-a-u_j}{(a+u_j)(a+u_i)}}$$

$$= \frac{x}{4(a+u_i)(a+u_j)}$$

Thus,

$$\partial_i \partial_j x = \frac{1}{2(u_i-u_j)}(\partial_j x - \partial_i x) .$$