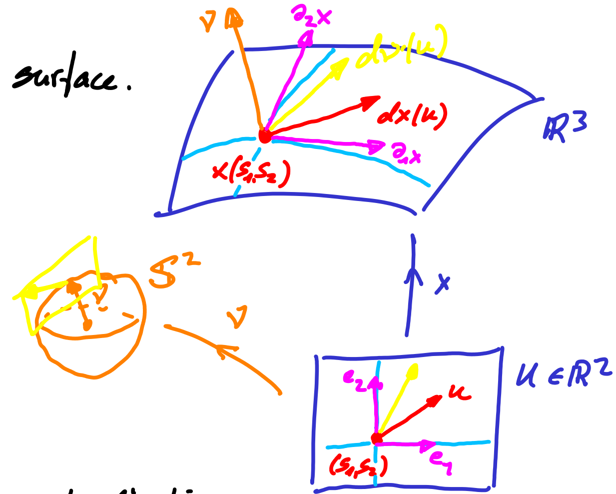
Let $x : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ smooth regular parametrized surface.



- x conjugate $\iff$ 2^{nd} fundamental form diagonal
  $\iff \langle \partial_1 x, \partial_2 \nu \rangle = \langle \partial_2 x, \partial_1 \nu \rangle = 0$
- x orthogonal $\iff$ 1^{st} fundamental form diagonal
  $\iff \langle \partial_1 x, \partial_2 x \rangle = 0$
- x curvature line parametrization
  $\iff$ the coordinate lines are along principal directions

Let $e_1, e_2 \in \mathbb{R}^2$ s.t. $dx(e_1) = \partial_1 x$, $dx(e_2) = \partial_2 x$. $\left[e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, dx = (\partial_1 x, \partial_2 x) \right.$

Then x is curvature line parametrization if and only if e_1, e_2 are eigenvectors of the shape operator S, i.e., $Se_1 = \kappa_1 e_1$, $Se_2 = \kappa_2 e_2$ (we assume $\kappa_1 \neq \kappa_2$)

where $\quad \mathrm{II}(u,v) = \mathrm{I}(u, Sv) = \mathrm{I}(Su, v)$.

$\Bigg[\mathrm{I}(u,v) = u^T \mathrm{I} v$ with $\mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$

$\mathrm{II}(u,v) = u^T \mathrm{II} v$ with $\mathrm{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix}$

$= \mathrm{I}(u, Sv) = u^T \mathrm{I} S v$

$\hookrightarrow \mathrm{II} = \mathrm{I}S \Rightarrow S = \mathrm{I}^{-1}\mathrm{II}$

Prop: x curvature line parametrization $\iff$ x orthogonal and conjugate

Proof: "$\Rightarrow$" $\langle \partial_1 x, \partial_2 x \rangle = \mathrm{I}(e_1, e_2) \overset{!}{=} 0$

Since S is self-adjoint w.r.t. I, its eigenvectors are orthogonal.

$\langle \partial_1 \nu, \partial_2 x \rangle = \mathrm{II}(e_1, e_2) = \mathrm{I}(e_1, Se_2) = \kappa_2 \mathrm{I}(e_1, e_2) = 0$

"$\Leftarrow$" $\begin{cases} \mathrm{I}(e_1, e_2) = 0 \\ \mathrm{II}(e_1, e_2) = \mathrm{I}(e_1, Se_2) = 0 \end{cases} \Rightarrow e_2, Se_2 \in e_1^{\perp} \Rightarrow Se_2 = \kappa_2 e_2$ for some $\kappa_2 \in \mathbb{R}$

Similarly for e_1.

Prop.: x curvature line parametrization $\iff$ x conjugate and $\begin{cases} \det(\nu, \partial_1 \nu, \partial_1 x) = 0 \\ \det(\nu, \partial_2 \nu, \partial_2 x) = 0 \end{cases}$

Note that $\det(\nu, \partial_i \nu, \partial_i x) = 0 \iff \partial_i \nu \sim \partial_i x$.

"$\Rightarrow$" $\langle \partial_1 \nu, \partial_2 x \rangle = 0$, $\langle \partial_1 x, \partial_2 x \rangle = 0 \Rightarrow \partial_1 \nu \sim \partial_1 x$

"$\Leftarrow$" $\langle \partial_1 \nu, \partial_2 x \rangle = 0$, $\partial_1 \nu \sim \partial_1 x \Rightarrow \langle \partial_1 x, \partial_2 x \rangle = 0$

In this case:

$$\kappa_1 = \frac{\mathrm{II}(e_1, e_1)}{\mathrm{I}(e_1, e_1)} = -\frac{\langle \partial_1 \nu, \partial_1 x \rangle}{\langle \partial_1 x, \partial_1 x \rangle}, \qquad \partial_1 \nu = \mu \partial_1 x$$

Thus,

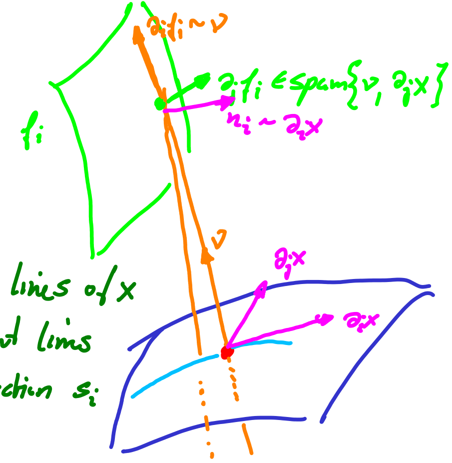
$$\langle \partial_1 \nu, \partial_1 x \rangle = \mu \langle \partial_1 x, \partial_1 x \rangle \Rightarrow \mu = -\kappa_1$$

and therefore

$$\begin{cases} \partial_1 \nu = -\kappa_1 \partial_1 x \\ \partial_2 \nu = -\kappa_2 \partial_2 x \end{cases}$$

Focal nets

$$f_i(s_1,s_2) = x(s_1,s_2) + \frac{1}{\kappa_i(s_1,s_2)}\nu(s_1,s_2) \qquad (\text{we assume } \kappa_i \neq 0)$$



regularity:

$$\partial_i f_i = \partial_i x + \partial_i\left(\frac{1}{\kappa_i}\right)\nu + \frac{1}{\kappa_i}\underbrace{\partial_i \nu}_{=-\kappa_i \partial_i x} = -\frac{\partial_i \kappa_i}{\kappa_i^2}\nu$$

⇝ the normal lines of x are the tangent lines of f_i in direction s_i

In particular, $\partial_i f_i = 0 \iff \partial_i \kappa_i = 0$

$$\partial_j f_i = \partial_j x + \partial_j\left(\frac{1}{\kappa_i}\right)\nu + \frac{1}{\kappa_i}\underbrace{\partial_j \nu}_{=-\kappa_j \partial_j x} = \left(1-\frac{\kappa_j}{\kappa_i}\right)\partial_j x - \frac{\partial_j \kappa_i}{\kappa_i^2}\nu$$

Thus, f_i is not regular if and only if $\partial_i \kappa_i = 0$ or $\underbrace{\left(1-\frac{\kappa_j}{\kappa_i}\right) = 0}_{\iff \kappa_i = \kappa_j \text{ (umbilic points)}}$

Prop: The focal nets are $\underbrace{\text{semi-geodesic}}$ conjugate nets

coordinate line in i-direction are geodesics on the surface f_i

Proof: conjugate:

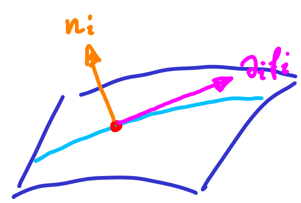
$$\partial_i \partial_j f_i = -\partial_j\left(\frac{\partial_i \kappa_i}{\kappa_i^2}\right)\nu - \frac{\partial_i \kappa_i}{\kappa_i^2}\underbrace{\partial_j \nu}_{=-\kappa_j \partial_j x} \in \mathrm{span}\{\nu, \partial_j x\}$$

which is the tangent plane of f_i

geodesic

If n_i is the normal vector field of f_i the we need to check

$$\partial_i^2 f_i \in \mathrm{span}\{\underbrace{n_i}_{\sim \partial_i x}, \underbrace{\partial_i f_i}_{\sim \nu}\} = \mathrm{span}\{\partial_i x, \nu\}$$

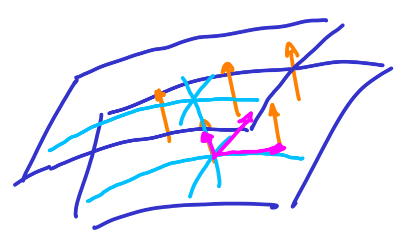


Indeed,

$$\partial_i^2 f_i = \partial_i\left(-\frac{\partial_i \kappa_i}{\kappa_i^2}\right)\nu - \frac{\partial_i \kappa_i}{\kappa_i^2}\underbrace{\partial_i \nu}_{\sim \partial_i x} \in \mathrm{span}\{\partial_i x, \nu\}$$

Parallel nets: Let x be a curvature line parametrization.

$$\tilde{x}(s_1,s_2,s_3) = x(s_1,s_2) + z(s_3)\nu(s_1,s_2)$$



regularity: $\partial_1 \tilde{x} = \partial_1 x + z\,\partial_1 \nu = (1 - z\kappa_1)\partial_1 x$

$\partial_2 \tilde{x} = (1 - z\kappa_2)\partial_2 x$

$\partial_3 \tilde{x} = z'\nu$

$$\det(\partial_1 \tilde{x}, \partial_2 \tilde{x}, \partial_3 \tilde{x}) = (1-z\kappa_1)(1-z\kappa_2)z'\underbrace{\det(\partial_1 x, \partial_2 x, \nu)}_{\neq 0}$$

$$= 0 \iff \underbrace{z = \frac{1}{\kappa_1} \text{ or } z = \frac{1}{\kappa_2}}_{\text{focal points of } x} \text{ or } z' = 0$$

orthogonality

$$\langle \partial_1 \tilde{x}, \partial_2 \tilde{x} \rangle = (1 - s\kappa_1)(1 - s\kappa_2) \underbrace{\langle \partial_1 x, \partial_2 x \rangle}_{=0} = 0$$

$$\langle \partial_1 \tilde{x}, \partial_3 \tilde{x} \rangle = (1 - s\kappa_1) s' \underbrace{\langle \partial_1 x, \nu \rangle}_{=0} = 0$$

$$\langle \partial_2 \tilde{x}, \partial_3 \tilde{x} \rangle = 0$$

3rd Lamé coefficient:

$$h_3^2 = \langle \partial_3 \tilde{x}, \partial_3 \tilde{x} \rangle = (s')^2 \underbrace{\langle \nu, \nu \rangle}_{=1} = (s')^2$$, which only depends on s_3