Lamé coefficients for 3D confocal quadrics

(and 1^{st} and 2^{nd} fundamental forms)

$$H_1^2 = \langle \partial_1 \mathbf{x}, \partial_1 \mathbf{x} \rangle = (\partial_1 x)^2 + (\partial_1 y)^2 + (\partial_1 z)^2$$

$$= \frac{1}{4}\left(\frac{x^2}{(u_1+a)^2} + \frac{y^2}{(u_1+b)^2} + \frac{z^2}{(u_1+c)^2} \right) = ?$$

$$x^2 = \frac{(u_1+a)(u_2+a)(u_3+a)}{(a-b)(a-c)} \Rightarrow 2x\partial_1 x = \partial_1(x^2) = \frac{x^2}{u_1+a} \Rightarrow \partial_1 x = \frac{x}{2(u_1+a)}$$

Lemma: For $-a < u_1 < -b < u_2 < -c < u_3$ and $\mathbf{x}(u_1,u_2,u_3) = \begin{pmatrix} x(u_1,u_2,u_3) \\ y(u_1,u_2,u_3) \\ z(u_1,u_2,u_3) \end{pmatrix}$

with $\frac{x^2}{u_i+a} + \frac{y^2}{u_i+b} + \frac{z^2}{u_i+c} = 1$, $i=1,2,3$.

Then
$$\begin{cases} \frac{x^2}{(u_1+a)^2} + \frac{y^2}{(u_1+b)^2} + \frac{z^2}{(u_1+c)^2} = \frac{(u_1-u_2)(u_1-u_3)}{(u_1+a)(u_1+b)(u_1+c)} \\ \frac{x^2}{(u_2+a)^2} + \frac{y^2}{(u_2+b)^2} + \frac{z^2}{(u_2+c)^2} = \frac{(u_2-u_1)(u_2-u_3)}{(u_2+a)(u_2+b)(u_2+c)} \\ \frac{x^2}{(u_3+a)^2} + \frac{y^2}{(u_3+b)^2} + \frac{z^2}{(u_3+c)^2} = \frac{(u_3-u_1)(u_3-u_2)}{(u_3+a)(u_3+b)(u_3+c)} \end{cases}.$$

Proof:

- u_1,u_2,u_3 are the roots of $f(\lambda) = \frac{x^2}{\lambda+a} + \frac{y^2}{\lambda+b} + \frac{z^2}{\lambda+c} - 1$

  $\hookrightarrow f(\lambda) = -\frac{(\lambda-u_1)(\lambda-u_2)(\lambda-u_3)}{(\lambda+a)(\lambda+b)(\lambda+c)}$

- differentiate both expressions for f w.r.t. u_i:

$$\frac{\partial_1(x^2)}{\lambda+a}+\frac{\partial_1(y^2)}{\lambda+b}+\frac{\partial_1(z^2)}{\lambda+c}=\frac{(\lambda-u_2)(\lambda-u_3)}{(\lambda+a)(\lambda+b)(\lambda+c)}$$

$$\overset{\partial_1(x^2)=\frac{x^2}{u_1+a}}{\Longleftrightarrow}\quad \frac{x^2}{(u_1+a)(\lambda+a)}+\frac{y^2}{(u_1+b)(\lambda+b)}+\frac{z^2}{(u_1+c)(\lambda+c)}= \quad - \, \text{"} \, -$$

$$\overset{\lambda=u_1}{\Longrightarrow}\quad \frac{x^2}{(u_1+a)^2}+\frac{y^2}{(u_1+b)^2}+\frac{z^2}{(u_1+c)^2}=\frac{(u_1-u_2)(u_1-u_3)}{(u_1+a)(u_1+b)(u_1+c)}\,.$$

$\square$

More generally: for $-a_1<u_1<-a_2<u_2<\dots<-a_N<u_N$ and $x(u_1,\dots,u_N)$

with $$\sum_{k=1}^{N}\frac{x_k^2}{u_i+a_k}=1\,,\quad i=1,\dots,N\,,$$

Then $$\sum_{k=1}^{N}\frac{x_k^2}{(u_i+a_k)^2}=\frac{\prod_{m\neq i}(u_i-u_m)}{\prod_{m=1}^{N}(u_i+a_m)}\,,\quad i=1,\dots,N\,.$$

Thus, the Lamé coefficients are given by:

$$H_1^2=\frac{1}{4}\frac{(u_1-u_2)(u_1-u_3)}{(u_1+a)(u_1+b)(u_1+c)}\,,\qquad H_2^2=\frac{1}{4}\frac{(u_2-u_1)(u_2-u_3)}{(u_2+a)(u_2+b)(u_2+c)}\,,$$

$$H_3^2=\frac{1}{4}\frac{(u_3-u_1)(u_3-u_2)}{(u_3+a)(u_3+b)(u_3+c)}$$

In particular,

$$\frac{E_{12}}{G_{12}}=\frac{H_1^2}{H_2^2}=-\frac{u_1-u_3}{(u_1+a)(u_1+b)(u_1+c)}\cdot\frac{(u_2+a)(u_2+b)(u_2+c)}{u_2-u_3}\,.\qquad I_{12}=\begin{pmatrix}E_{12}&0\\0&G_{12}\end{pmatrix}$$

Generally,

$$\frac{E_{ij}}{G_{ij}}=\frac{H_i^2}{H_j^2}=-\frac{\alpha_{jk}(u_j,u_k)}{\alpha_{ik}(u_i,u_k)}\quad\text{with}\quad \alpha_{lm}(u_l,u_m)=\frac{(u_l+a)(u_l+b)(u_l+c)}{u_l-u_m}\,,$$

and thus, all coordinate surfaces (i.e. quadrics) are isothermic surfaces
(but we can't reparametrize the entire confocal system in such a way that all

coordinate surfaces are parametrized isothermically at the same time)

Let us now compute the 2^{nd} fundamental forms of the coordinate surfaces:

$$e_{12} = \frac{\partial_3 (H_1^2)}{H_3} = -\frac{1}{4H_3} \cdot \frac{u_1 - u_2}{(u_1+a)(u_1+b)(u_1+c)}$$

$$\mathrm{II}_{12} = \begin{pmatrix} e_{12} & 0 \\ 0 & g_{12} \end{pmatrix}$$

$$g_{12} = \frac{\partial_3 (H_2^2)}{H_3} = -\frac{1}{4H_3} \cdot \frac{u_2 - u_1}{(u_2+a)(u_2+b)(u_2+c)}$$

Where are the umbilic points on the quadrics (coordinate surfaces)?

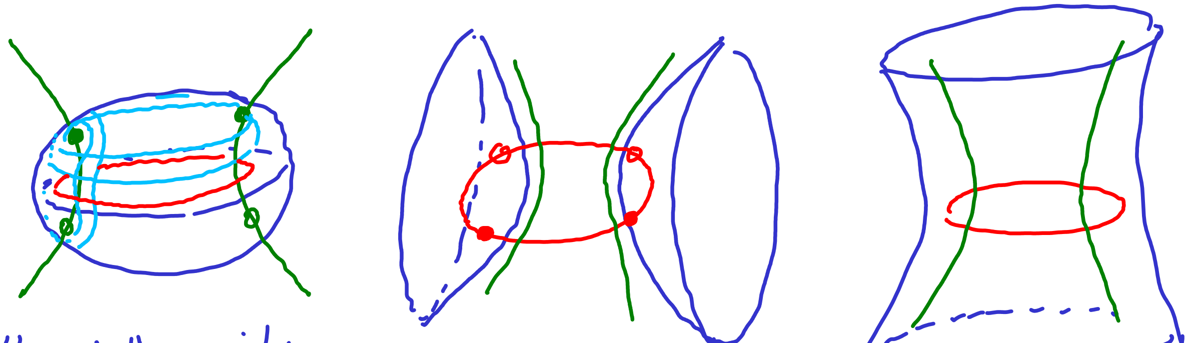
$$\kappa_1 = \frac{e_{12}}{E_{12}} = -\frac{1}{4H_3} \cdot \frac{u_1 - u_2}{(u_1+a)(u_1+b)(u_1+c)} \cdot \frac{4(u_1+a)(u_1+b)(u_1+c)}{(u_1-u_2)(u_1-u_3)} = \frac{-1}{H_3(u_1-u_3)}$$

$$\kappa_2 = \frac{g_{12}}{G_{12}} = -\frac{1}{4H_3} \cdot \frac{u_2 - u_1}{(u_2+a)(u_2+b)(u_2+c)} \cdot \frac{4(u_2+a)(u_2+b)(u_2+c)}{(u_2-u_1)(u_2-u_3)} = \frac{-1}{H_3(u_2-u_3)}$$

$$\kappa_1 = \kappa_2 \iff \frac{-1}{H_3(u_1-u_3)} = \frac{-1}{H_3(u_2-u_3)} \iff u_1 = u_2 \iff u_1 = u_2 = -b$$

(point on the focal hyperbola)

Generally: The umbilic points are the intersection points of the quadric with its focal conics.



4 umbilic points   4 umbilic points

Let us now have a look at the quotients of the coefficients of the 2^{nd} fundamental form:

$$\frac{e_{12}}{g_{12}} = -\frac{(u_2+a)(u_2+b)(u_2+c)}{(u_1+a)(u_1+b)(u_1+c)}$$

Generally,

$$\frac{e_{ij}}{g_{ij}} = -\frac{\beta_j(u_j)}{\beta_i(u_i)}, \qquad \beta_i(u_i) = (u_i+a)(u_i+b)(u_i+c).$$

How do the coefficients of the 2^{nd} fundamental form change under reparametrization along the coordinate lines $u_i \longrightarrow u_i(s_i)$

$$\tilde{e}_{12}(s_1,s_2) = \left\langle \frac{\partial \nu}{\partial s_1}, \frac{\partial x}{\partial s_1} \right\rangle = \left\langle \frac{du_1}{ds_1}\frac{\partial \nu}{\partial u_1}, \frac{du_1}{ds_1}\frac{\partial x}{\partial u_1} \right\rangle = u_1'(s_1)^2 \left\langle \frac{\partial \nu}{\partial u_1}, \frac{\partial x}{\partial u_1} \right\rangle$$
$$= u_1'(s_1)^2\, e_{12}\big(u_1(s_1), u_2(s_2)\big)$$

Thus, the quotient changes by

$$\frac{\tilde{e}_{12}}{\tilde{g}_{12}} = \frac{(u_1')^2 e_{12}}{(u_2')^2 g_{12}} = -\frac{(u_1')^2 \overbrace{\overbrace{(u_2+a)}^{>0}\overbrace{(u_2+b)}^{>0}\overbrace{(u_2+c)}^{<0}}^{<0}}{(u_2')^2 \underbrace{\underbrace{(u_1+a)}_{>0}\underbrace{(u_1+b)}_{<0}\underbrace{(u_1+c)}_{<0}}_{>0}} \qquad \left[-a < u_1 < -b < u_2 < -c < u_3\right.$$

If we choose

$$\left.\begin{aligned} (u_1')^2 &= (u_1+a)(u_1+b)(u_1+c) > 0 \\ (u_2')^2 &= -(u_2+a)(u_2+b)(u_2+c) > 0 \\ (u_3')^2 &= (u_3+a)(u_3+b)(u_3+c) > 0 \end{aligned}\right\}$$

differential equations for Weierstraß-$\wp$-function (in particular they can be solved by elliptic functions)

we obtain

$$\frac{\tilde{e}_{12}}{\tilde{g}_{12}} = 1, \quad \frac{\tilde{e}_{13}}{\tilde{g}_{13}} = -1, \quad \frac{\tilde{e}_{23}}{\tilde{g}_{23}} = 1,$$

thus up to sign we can make all 2^{nd} fundamental forms of the coordinate surfaces "conformal". (the parametrization we derived using Jacobi elliptic functions has exactly this property)