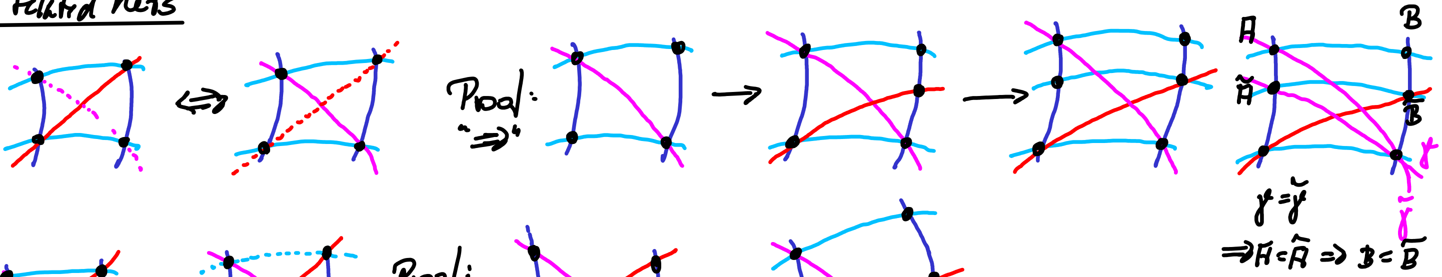
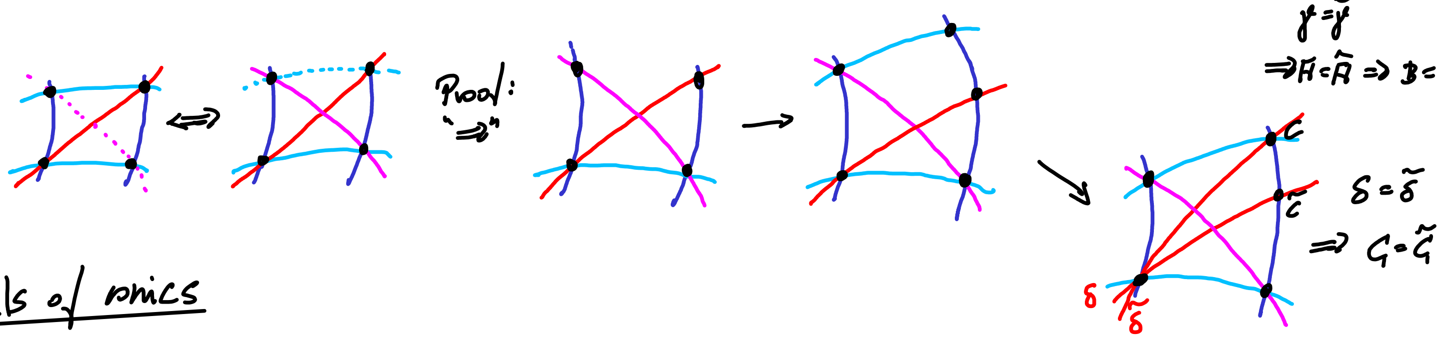
Diagonally related nets

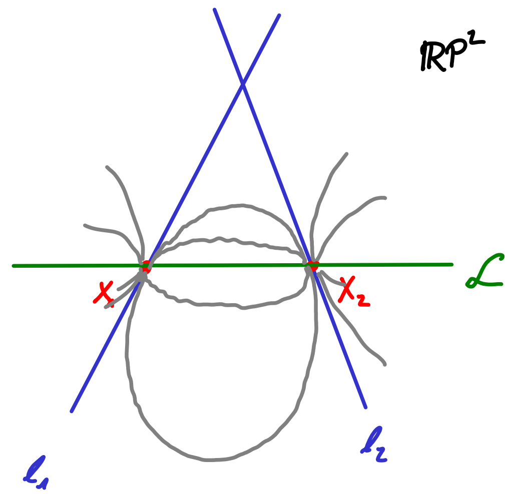
Show: 

Show: 

Pencils of conics

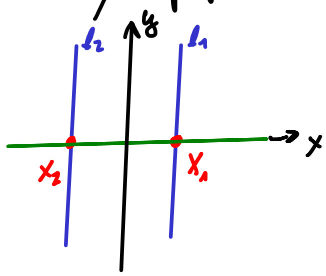
Lemma: ℓ_1, ℓ_2 two lines in $\mathbb{RP}^2$,
$X_1 \in \ell_1$, $X_2 \in \ell_2$ two points

Then: family of conics tangent to ℓ_1 in X_1 and ℓ_2 in X_2
= pencil of conics spanned by $\mathcal{L}$ and $\mathcal{D}$.



$\mathcal{D} = \ell_1 \cup \ell_2$

Proof: Normalize by a projective transformation:



$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\mathcal{D}: (x_1 - x_3)(x_1 + x_3) = x_1^2 - x_3^2 = 0 \longleftrightarrow \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$

$\ell_1: x = 1 \iff x_1 - x_3 = 0 \iff \ell_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}^*$

$\mathcal{L}: x_2^2 = 0 \longleftrightarrow \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}$

$\ell_2: x = -1 \iff x_1 + x_3 = 0 \iff \ell_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^*$

Start with an arbitrary conic $Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ & q_{22} & q_{23} \\ & & q_{33} \end{pmatrix}$

X_1 lies on Q: $(1\ 0\ 1)\, Q \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (1\ 0\ 1) \begin{pmatrix} q_{11} + q_{13} \\ q_{12} + q_{23} \\ q_{13} + q_{33} \end{pmatrix} = q_{11} + q_{33} + 2q_{13} = 0$

X_2 lies on Q: $(-1\ 0\ 1)\, Q \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = q_{11} + q_{33} - 2q_{13} = 0$

Q tangent to ℓ_1 in X_1 $\iff$ ℓ_1 polar line of X_1: $Q \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \iff \begin{cases} q_{12} + q_{23} = 0 \\ q_{11} + q_{13} = -(q_{13} + q_{33}) \end{cases}$
$\iff q_{11} + q_{33} + 2q_{13} = 0$

Q tangent to ℓ_2 in X_2: $Q \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \iff \begin{cases} -q_{12} + q_{23} = 0 \\ -q_{11} + q_{13} = -q_{13} + q_{33} \iff q_{11} + q_{33} - 2q_{13} \end{cases}$

Note: $X_1 \in \ell_1$ and ℓ_1 polar of $X_1 \Rightarrow X_1 \in Q$. That is why $X_1 \in Q$ leads to a redundant equation.
Same for $X_2 \in Q$.

Thus we obtain 4 linear equations:

$$\left.\begin{array}{l} q_{12}+q_{23}=0 \\ q_{12}-q_{23}=0 \end{array}\right\} \iff \left\{\begin{array}{l} q_{12}=0 \\ q_{23}=0 \end{array}\right.$$

$$\left.\begin{array}{l} q_{11}+q_{33}+2q_{13}=0 \\ q_{11}+q_{33}-2q_{13}=0 \end{array}\right\} \iff \left\{\begin{array}{l} q_{13}=0 \\ q_{11}+q_{33}=0 \end{array}\right.$$

$$\iff Q=\begin{pmatrix} \lambda & & \\ & \mu & \\ & & -\lambda \end{pmatrix} \text{ for some } \lambda,\mu$$

$$=\lambda\underbrace{\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}}_{\mathcal{D}}+\mu\underbrace{\begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}}_{\mathcal{L}}$$