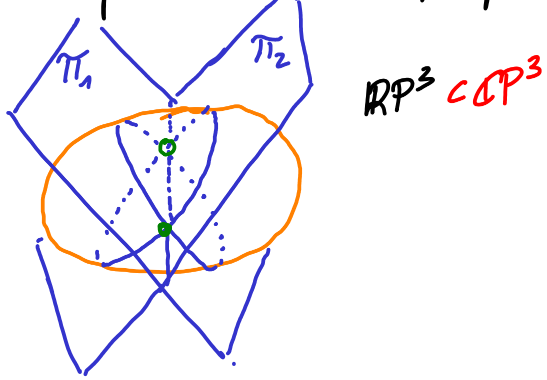
circles and spheres

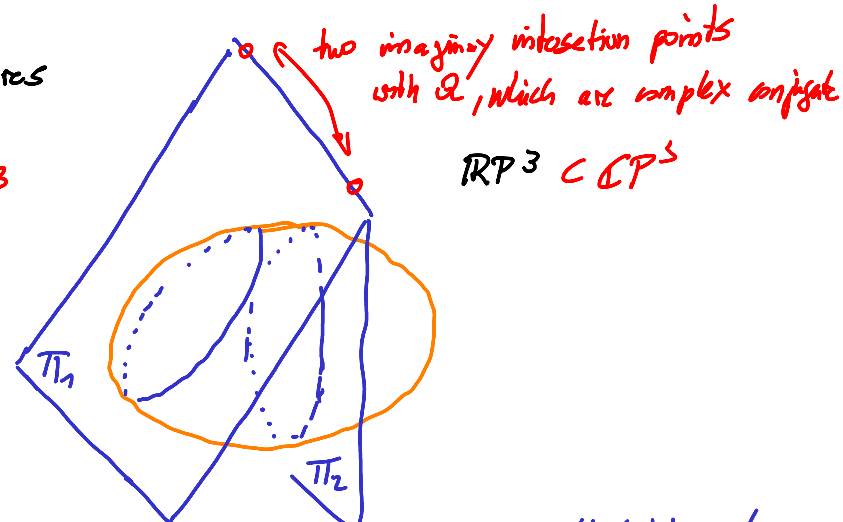
- $Q \subset \mathbb{RP}^3 \subset \mathbb{CP}^3$ quadric.

  Q sphere $\iff Z \subset Q$ where $Z: x_1^2 + x_2^2 + x_3^2 = 0,\ x_4 = 0$

- circles are planar sections of spheres
- generally



two planes Π_1, Π_2 that intersect in a line, which intersects Q in 2 points



two planes Π_1, Π_2 that intersect in a line, which doesn't intersect Q

in $\mathbb{CP}^3$ these two cases coincide:

- a line always has two intersection points (or one double intersection point) with a quadric
- if the intersection points are not real, they must be complex conjugate (and this case there are always two intersection points)

- thus, for a sphere, a planar section must intersect Z in two complex conjugate points

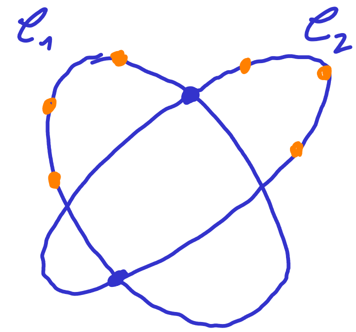
  this gives us: $\mathcal{C}$ conic in some plane in $\mathbb{RP}^3$

  "$\Rightarrow$" $\mathcal{C}$ circle $\Rightarrow$ $\mathcal{C}$ contains two points of Z

- "$\Leftarrow$" if $\mathcal{C}$ contains two points of Z,

  we need to show that there exists a quadric that contains $\mathcal{C}$ and Z

for two conics that intersect in two points there always exists a pencil of quadrics containing the two conics:



choose 3 points on each conic

$\hookrightarrow$ 8 points determine a pencil of quadrics, each quadric containing $\mathcal{C}_1$ and $\mathcal{C}_2$

- alternative approach by normalizing the plane of the circle

  $\mathcal{C} \subset \Pi$ conic in $\mathbb{R}P^3$

apply a similarity transformation (projective transformation that preserves $\mathcal{Z}$)

to normalize Π, e.g. $\Pi : x_3 = 0$

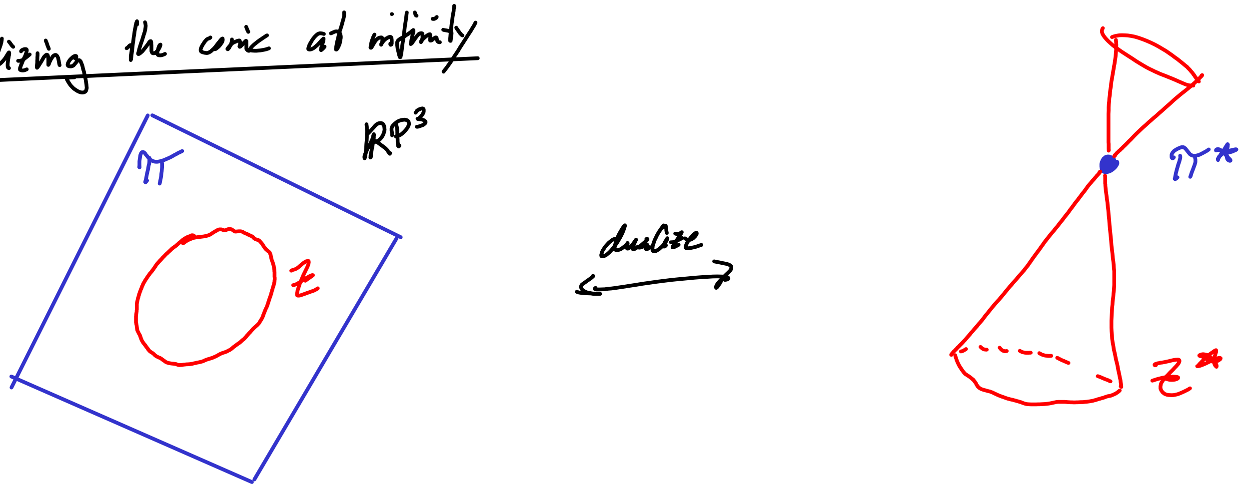
restrict the problem to the plane Π:

$\mathcal{C}$: quadratic equation in 3 variables x_1, x_2, x_4

$\mathcal{Z}$: $x_1^2 + x_2^2 = 0$, $x_4 = 0$ (two imaginary points at infinity)

show $\mathcal{C}$ is a circle if and only if it contains the 2 points on $\mathcal{Z}$

dualizing the conic at infinity



$\mathcal{Z} : x_1^2 + x_2^2 + x_3^2 = 0, \; x_4 = 0$

$\mathcal{Z}^* : x_1^2 + x_2^2 + x_3^2 = 0$

Gram matrix: $Z = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$

circular sections of ellipsoids

$$Q: \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \, , \quad a > b > c > 0 \qquad \text{ellipsoid}$$

circular sections lie in the planes:

$$(1) \quad \Pi_\pm(\lambda_\pm): \sqrt{\frac{1}{b} - \frac{1}{a}}\, x \pm \sqrt{\frac{1}{c} - \frac{1}{b}}\, z = \lambda_\pm \, , \qquad \lambda_\pm \in \left[-\sqrt{\frac{a-c}{b}}, \sqrt{\frac{a-c}{b}} \right]$$

curvature line parametrization of Q:

$$(2) \quad \begin{cases} x(s_1,s_2) = \sqrt{\dfrac{a}{(a-b)(a-c)}}\, f_1(s_1) f_2(s_2) \\ y(s_1,s_2) = \sqrt{\dfrac{b}{(a-b)(b-c)}}\, g_1(s_1)\, g_2(s_2) \\ z(s_1,s_2) = \sqrt{\dfrac{c}{(a-c)(b-c)}}\, h_1(s_1)\, h_2(s_2) \end{cases}$$

plus some quadratic equations for $f_1, f_2, g_1, g_2, h_1, h_2$

substitute (2) into (1):

$$\sqrt{\frac{\cancel{a}}{(\cancel{a-b})(a-c)} \cdot \frac{\cancel{a-b}}{\cancel{a}b}}\, f_1(s_1) f_2(s_2) \pm \sqrt{\frac{\cancel{c}}{(a-c)(\cancel{b-c})} \frac{\cancel{b-c}}{b\cancel{c}}}\, h_1(s_1) h_2(s_2) = \lambda_\pm$$

$$\iff f_1(s_1) f_2(s_2) \pm h_1(s_1) h_2(s_2) = \sqrt{b(a-c)}\, \lambda_\pm$$

Thus, f_1, f_2, h_1, h_2 need to satisfy

$$(\mathrm{I}) \qquad f_1(s_1)^2 + h_1(s_1)^2 = a - c$$

$$(\mathrm{II}) \qquad f_2(s_2)^2 + h_2(s_2)^2 = a - c$$

$$(\mathrm{III}_\pm) \qquad f_1(s_1) f_2(s_2) \pm h_1(s_1) h_2(s_2) = \sqrt{b(a-c)}\, \lambda_\pm(s_1 \pm s_2) \qquad \text{for some functions } \lambda_\pm(s_\pm)$$

Check that $f_1(s_1) = \sqrt{a-c}\,\sin s_1$ $\qquad f_2(s_2) = \sqrt{a-c}\,\cos s_2$

$h_1(s_1) = \sqrt{a-c}\,\cos s_1$ $\qquad h_2(s_2) = \sqrt{a-c}\,\sin s_2$

satisfy (I), (II), (III$\pm$):

(I) $(a-c)\sin^2 s_1 + (a-c)\cos^2 s_1 = a-c$ ✓

(II) $(a-c)\cos^2 s_2 + (a-c)\sin^2 s_2 = a-c$ ✓

(III$_\pm$) $(a-c)\sin s_1 \cos s_2 \pm (a-c)\cos s_1 \sin s_2$

$$= (a-c)\Big(\underbrace{\sin s_1 \cos s_2 \pm \cos s_1 \sin s_2}_{\sin(s_1 \pm s_2)}\Big)$$

$$= \sqrt{b(a-c)} \cdot \underbrace{\sqrt{\frac{a-c}{b}}\,\sin(s_1 \pm s_2)}_{= \lambda_\pm(s_1 \pm s_2)}$$ ✓

in the discrete case:

- the planes are given by (1)
- the curvature line parametrization is given (2) with $s_1 \to n_1$, $s_2 \to n_2$ with different quadratic eq. for $f_1, f_2, g_1, g_2, h_1, h_2$

In particular, f_1, f_2, h_1, h_2 must satisfy:

(I) $f_1(n_1) f_1(n_1 + \frac{1}{2}) + h_1(n_1) h_1(n_1 + \frac{1}{2}) = a-c$

(II) $f_2(n_2) f_2(n_2 + \frac{1}{2}) + h_2(n_2) h_2(n_2 + \frac{1}{2}) = a-c$

(III$_\pm$) $f_1(n_1) f_2(n_2) + h_1(n_1) h_2(n_2) = \sqrt{b(a-c)}\, \lambda_\pm\left(\frac{n_1 \pm n_2}{2}\right)$

Check that $f_1(n_1) = \varepsilon\sqrt{a-c}\,\sin \delta n_1$, $\quad f_2(n_2) = \varepsilon\sqrt{a-c}\,\cos \delta n_2$

$h_1(n_1) = \varepsilon\sqrt{a-c}\,\cos \delta n_1$, $\quad h_2(n_2) = \varepsilon\sqrt{a-c}\,\sin \delta n_2$

for $0 < \delta < \pi$ and $\varepsilon = \frac{1}{\sqrt{\cos\frac{\delta}{2}}}$

satisfy (I), (II), (III$\pm$):

(I) $\quad \varepsilon^2(a-c)\sin\delta n_1 \sin\left(\delta n_1+\frac{\delta}{2}\right) + \varepsilon^2(a-c)\cos\delta n_1 \cos\left(\delta n_1+\frac{\delta}{2}\right)$

$$= \varepsilon^2(a-c)\Big(\underbrace{\sin\delta n_1 \sin\left(\delta n_1+\tfrac{\delta}{2}\right) + \cos\delta n_1 \cos\left(\delta n_1+\tfrac{\delta}{2}\right)}_{=\cos\left(\delta n_1 - \left(\delta n_1+\frac{\delta}{2}\right)\right)}\Big)$$

$$= \varepsilon^2(a-c)\cos\frac{\delta}{2} = a-c \quad ✓$$

(II) similar to (I) ✓

(III±) $\quad \varepsilon^2(a-c)\sin\delta n_1 \cos\delta n_2 \pm \varepsilon^2(a-c)\cos\delta n_1 \sin\delta n_2$

$$= \varepsilon^2(a-c)\sin\big(\delta(n_1\pm n_2)\big)$$

$$= \sqrt{b(a-c)}\,\underbrace{\sqrt{\frac{a-c}{b}}\cdot\varepsilon^2 \sin\big(\delta(n_1\pm n_2)\big)}_{=\lambda_\pm\left(\frac{n_1\pm n_2}{2}\right)} \quad ✓$$