Polynomial matrices

Exercise:
$$A = A_s\lambda^s + A_{s-1}\lambda^{s-1} + \dots + A_0 \in \mathbb{F}[\lambda]^{n\times n}\setminus\{0\};\ \deg A = s,\ \deg B = t$$
$$B = B_t\lambda^t + B_{t-1}\lambda^{t-1} + \dots + B_0 \qquad A_s \neq 0,\ B_t \neq 0$$

(i) Assume A_s is invertible. Then $A_sB_t \neq 0$
(Otherwise, if $A_sB_t = 0$, then $B_t = 0$ ↯)

$$AB = A_sB_t\lambda^{s+t} + \dots$$
Thus, $\deg(AB) = \deg A + \deg B$.

(ii)
$$A = \begin{pmatrix} \lambda & 0 \\ \lambda^2 & \lambda+1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{not invertible}}\lambda^2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\lambda + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \lambda^2 & 0 \\ 2\lambda^3+\lambda^2 & (\lambda+1)^2 \end{pmatrix}.$$ So $\deg A^2 = 3$

Prop. (division of polynomial matrices)
Let $A, B \in \mathbb{F}[\lambda]^{n\times n}$, leading coefficient matrix of B is invertible.
Then $\exists!\ Q, R \in \mathbb{F}[\lambda]^{n\times n}$: $A = QB + R$, $\deg R < \deg B$

Uniqueness: Let $A = QB + R = \tilde{Q}B + \tilde{R}$, $\deg R < \deg B$, $\deg \tilde{R} < \deg B$.
$$\Rightarrow (Q - \tilde{Q})B = \tilde{R} - R$$

Assume $Q \neq \tilde{Q}$.
Then
$$\deg B \leq \underbrace{\deg\big((Q-\tilde{Q})B\big)}_{=\ \deg(Q-\tilde{Q}) + \deg B} = \deg(\tilde{R} - R) < \deg B \ ↯$$

Thus, $Q = \tilde{Q}$ and $R = \tilde{R}$.

Lemma: Interchanging two rows (or two columns) is a sequence of elementary operations.

$$\underbrace{\begin{pmatrix} \vdots \\ -\ a\ - \\ \vdots \\ -\ b\ - \\ \vdots \end{pmatrix}}_{\in \mathbb{F}[\lambda]^{n\times n},\ a,b \in \mathbb{F}[\lambda]^{1\times n}} \longrightarrow \begin{pmatrix} \vdots \\ a \\ \vdots \\ a+b \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots \\ -b \\ \vdots \\ a+b \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots \\ -b \\ \vdots \\ a \\ \vdots \end{pmatrix} \longrightarrow \begin{pmatrix} \vdots \\ b \\ \vdots \\ a \\ \vdots \end{pmatrix}$$

$$\left[\ \text{elementary matrices: } \underset{\alpha \in \mathbb{F}\setminus\{0\}}{\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \alpha & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}},\quad \underset{a \in \mathbb{F}[\lambda]}{\begin{pmatrix} 1 & & & a \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}}\ \right]$$

Smith normal form

Example:

$$A = \begin{pmatrix} 0 & 0 & \lambda^2-1 \\ \lambda^2+\lambda & \lambda^2-\lambda & 0 \\ \boxed{\lambda^2} & 0 & 0 \end{pmatrix}, \quad \deg A = 2$$

$$\delta(A) = \min\{\deg a_{ij} \mid a_{ij} \neq 0\} = 2$$

(i) Choose a_{ij} with $\deg a_{ij} = \delta(A)$. Make it monic. Bring it to position a_{11}.

$$\longrightarrow \begin{pmatrix} \lambda^2 & 0 & 0 \\ \lambda^2+\lambda & \lambda^2-\lambda & 0 \\ 0 & 0 & \lambda^2-1 \end{pmatrix}$$

(ii) Decrease degree along the 1st column.

$$\longrightarrow \begin{pmatrix} \lambda^2 & 0 & 0 \\ \lambda & \lambda^2-\lambda & 0 \\ 0 & 0 & \lambda^2-1 \end{pmatrix}, \quad \delta(A) = 1 \rightsquigarrow \text{go to (i)}$$

(i)

$$\longrightarrow \begin{pmatrix} \lambda & \lambda^2-\lambda & 0 \\ \lambda^2 & 0 & 0 \\ 0 & 0 & \lambda^2-1 \end{pmatrix}$$

(ii)

$$\longrightarrow \begin{pmatrix} \lambda & \lambda^2-\lambda & 0 \\ 0 & -\lambda^3+\lambda^2 & 0 \\ 0 & 0 & \lambda^2-1 \end{pmatrix}, \quad \delta(A)=1 \rightsquigarrow \text{go to (iii)}$$

(iii) Decrease degree along the 1st row.

$$\longrightarrow \begin{pmatrix} \overset{a_{11}}{\boxed{\lambda}} & 0 & 0 \\ 0 & -\lambda^3+\lambda^2 & 0 \\ 0 & 0 & \lambda^2-1 \end{pmatrix}, \quad \delta(A)=1 \rightsquigarrow \text{go to (iv)}$$

$\tilde{A}$

(iv) Is every entry in $\tilde{A}$ divisible by a_{11}?

No, λ^2-1 is not divisible by λ. $\rightsquigarrow$ Decrease degree of λ^2-1

$$\longrightarrow \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda^3+\lambda^2 & 0 \\ \lambda & 0 & \lambda^2-1 \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda & 0 & -\lambda^2 \\ 0 & -\lambda^3+\lambda^2 & 0 \\ \lambda & 0 & -1 \end{pmatrix}, \quad \delta(A)=0 \rightsquigarrow \text{go to (i)}$$

(i)

$$\longrightarrow \begin{pmatrix} 1 & 0 & \lambda \\ 0 & -\lambda^3+\lambda^2 & 0 \\ \lambda^2 & 0 & \lambda \end{pmatrix}$$

(ii)

$$\longrightarrow \begin{pmatrix} 1 & 0 & \lambda \\ 0 & -\lambda^3+\lambda^2 & 0 \\ 0 & 0 & -\lambda^3+\lambda \end{pmatrix}$$

(iii)

$$\longrightarrow \begin{pmatrix} \overset{a_{11}}{\boxed{1}} & 0 & 0 \\ 0 & -\lambda^3+\lambda^2 & 0 \\ 0 & 0 & -\lambda^3+\lambda \end{pmatrix}$$

$\tilde{A}$

(iv) Yes, every entry of $\tilde{A}$ is divisible by a_{11}. $\rightsquigarrow$ continue with $\tilde{A}$ at (i)

$$\tilde{A} = \begin{pmatrix} -\lambda^3+\lambda^2 & 0 \\ 0 & -\lambda^3+\lambda \end{pmatrix}, \quad \delta(\tilde{A}) = 3$$

(i)

$$\longrightarrow \begin{pmatrix} \overset{a_{11}}{\boxed{\lambda^3-\lambda^2}} & 0 \\ 0 & \underset{\tilde{A}}{\boxed{-\lambda^3+\lambda}} \end{pmatrix}$$

(ii) (iii) already done.

(iv) $\tilde{A}$ not divisible by a_{11}.

$$\longrightarrow \begin{pmatrix} \lambda^3-\lambda^2 & 0 \\ \lambda^3-\lambda^2 & -\lambda^3+\lambda \end{pmatrix} \longrightarrow \begin{pmatrix} \lambda^3-\lambda^2 & \lambda^3-\lambda^2 \\ \lambda^3-\lambda^2 & -\lambda^2+\lambda \end{pmatrix}, \quad \delta(A)=2$$

(i)

$$\longrightarrow \begin{pmatrix} \lambda^2-\lambda & \lambda^3-\lambda^2 \\ -\lambda^3+\lambda^2 & \lambda^3-\lambda^2 \end{pmatrix}$$

(ii)

$$\longrightarrow \begin{pmatrix} \lambda^2-\lambda & \lambda^3-\lambda^2 \\ 0 & \lambda^4-\lambda^2 \end{pmatrix}$$

(iii)

$$\longrightarrow \begin{pmatrix} \overset{a_{11}}{\boxed{\lambda^2-\lambda}} & 0 \\ 0 & \underset{\tilde{A}}{\boxed{\lambda^4-\lambda^2}} \end{pmatrix} = \begin{pmatrix} \lambda(\lambda-1) & 0 \\ 0 & \lambda^2(\lambda-1)(\lambda+1) \end{pmatrix}$$

(iv) $\tilde{A}$ is divisible by a_{11} $\rightsquigarrow$ $\boxed{\text{done}}$

↳ A is equivalent to

$$\begin{pmatrix} 1 & & \\ & \lambda(\lambda-1) & \\ & & \lambda^2(\lambda-1)(\lambda+1) \end{pmatrix}.$$