minor divisors and invariant factors

$A \in \mathbb{F}[\lambda]^{n\times n}$, $\quad D_k$ = monic greatest common divisor of all $k\times k$ minors of A

Lemma: D_k invariant under elementary operations

1) $A = \begin{pmatrix} \vdots \\ p \\ \vdots \end{pmatrix}$, $p \in \mathbb{F}[\lambda]^{1\times n}$

$$\xrightarrow[\text{of } 1^{\text{st}} \text{ type}]{\text{elem. op.}} \begin{pmatrix} \vdots \\ \alpha p \\ \vdots \end{pmatrix} \quad \text{with } \alpha \in \mathbb{F}\setminus\{0\}$$

m some minor of A

case 1: the row p is not contained in the minor m

$$m = \det \begin{pmatrix} \vdots \\ p \\ \vdots \end{pmatrix} \xrightarrow{\text{elem. op.}} m$$

case 2: the row p is contained in the minor

$$m = \det \begin{pmatrix} \vdots \\ p \\ \vdots \end{pmatrix} \longrightarrow \det \begin{pmatrix} \vdots \\ \alpha p \\ \vdots \end{pmatrix} = \alpha m$$

2) $A = \begin{pmatrix} \vdots \\ p \\ \vdots \\ q \\ \vdots \end{pmatrix}$, $p, q \in \mathbb{F}[\lambda]^{1\times n}$

$$\xrightarrow[\text{of } 2^{\text{nd}} \text{ type}]{\text{elem. op.}} \begin{pmatrix} \vdots \\ p \\ \vdots \\ q + ap \\ \vdots \end{pmatrix} \quad \text{with some } a \in \mathbb{F}[\lambda]$$

case 1: the row q is not contained in the minor

$$m = \det \begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \det \begin{pmatrix} p \\ q + ap \end{pmatrix} = m$$

case 2: the row p and q are contained in the minor

$$m = \det\begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \det\begin{pmatrix} p \\ q+ap \end{pmatrix} = \underbrace{\det\begin{pmatrix} p \\ q \end{pmatrix}}_{=m} + a\,\underbrace{\det\begin{pmatrix} p \\ p \end{pmatrix}}_{=0}$$

case 3: the row q is contained, but not the row p

$$m = \det\begin{pmatrix} \cancel{p} \\ q \end{pmatrix} \longrightarrow \det\begin{pmatrix} \cancel{p} \\ q+ap \end{pmatrix} = \underbrace{\det\begin{pmatrix} \cancel{p} \\ q \end{pmatrix}}_{=m} + a\,\det\begin{pmatrix} \cancel{p} \\ p \end{pmatrix}$$

|| up to sign

$$\underbrace{\det\begin{pmatrix} p \\ \cancel{q} \end{pmatrix}}_{\substack{= k\times k \text{ minor of } A \\ =: \tilde m}}$$

$$= m + \tilde a \tilde m$$

Example

$$A = \begin{pmatrix} 0 & 0 & \lambda^2-1 \\ \lambda^2+\lambda & \lambda^2-\lambda & 0 \\ \lambda^2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & (\lambda-1)(\lambda+1) \\ \lambda(\lambda+1) & \lambda(\lambda-1) & 0 \\ \lambda^2 & 0 & 0 \end{pmatrix} \quad \mathcal{D}_0$$

$= \lambda^0(\lambda-1)^0(\lambda+1)^0$

1×1 minors: $(\lambda-1)(\lambda+1)$, $\lambda(\lambda+1)$, $\lambda(\lambda-1)$, λ^2 $\rightsquigarrow \mathcal{D}_1 = 1$ $= \lambda^0(\lambda-1)^0(\lambda+1)^0$

2×2 minors: $-\lambda^3(\lambda-1)$, $-\lambda^2(\lambda-1)(\lambda+1)$, $-\lambda(\lambda-1)^2(\lambda+1)$ $\rightsquigarrow \mathcal{D}_2 = \lambda(\lambda-1) = \lambda^1(\lambda-1)^1(\lambda+1)^0$

3×3 minors: $-\lambda^3(\lambda-1)^2(\lambda+1)$ $\rightsquigarrow \mathcal{D}_3 = \lambda^3(\lambda-1)^2(\lambda+1)$ $= \lambda^3(\lambda-1)^2(\lambda+1)^1$

invariant factors:

$$I_1 = \frac{\mathcal{D}_1}{\mathcal{D}_0} = 1 = \lambda^0(\lambda-1)^0(\lambda+1)^0$$

$$I_2 = \frac{\mathcal{D}_2}{\mathcal{D}_1} = \lambda(\lambda-1) = \lambda^1(\lambda-1)^1(\lambda+1)^0$$

$$I_3 = \frac{\mathcal{D}_3}{\mathcal{D}_2} = \lambda^2(\lambda-1)(\lambda+1) = \lambda^2(\lambda-1)^1(\lambda+1)^1$$

Smith normal form:

$$\begin{pmatrix} 1 & & \\ & \lambda(\lambda-1) & \\ & & \lambda^2(\lambda-1)(\lambda+1) \end{pmatrix}$$

same as last time when computed by the algorithm list of elementary divisors: $\lambda^2, \lambda, \lambda-1, \lambda-1, \lambda+1$

reconstruct invariant factors

segre symbol: $[(0:2,1),(1:1\,1),(-1:1)]$

or abbreviated: $[\underbrace{(2,1)}_{\text{2 entries}},\underbrace{(1,1)}_{\text{2 entries}},\underbrace{1}_{\text{1 entry}}]$

$I_3 = \underbrace{\lambda^2(\lambda-1)(\lambda+1)}$

all elementary divisors with largest exponent (and erase them from the list)

$I_2 = \underbrace{\lambda(\lambda-1)}$

all elementary divisors with largest exponent that are still left

$I_1 = 1$

degenerate matrices in the family $A(\lambda)$

$$A(0) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{rk} A(0) = 1 = 3-2$$

$$A(1) = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \operatorname{rk} A(1) = 1 = 3-2$$

$$A(-1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \operatorname{rk} A(-1) = 2 = 3-1$$