Segre symbols for polynomial matrices of degree 1

For $A \in \mathbb{C}[\lambda]^{n\times n}$, $\deg A = 1$, $A = A_1\lambda + A_0$, $A_1, A_0 \in \mathbb{C}^{n\times n}$, $\det A_1 \neq 0$

then necessary and sufficient conditions on the Segre symbol

$$\left[(\lambda_1 : \nu_{11}, \dots, \nu_{1h_1}), \dots, (\lambda_s : \nu_{s1}, \dots, \nu_{sh_s})\right]$$

of A are given by

(i) $1 \leq s \leq n$

(ii) $\lambda_1, \dots, \lambda_s \in \mathbb{C}$, $\lambda_i \neq \lambda_j$ for $i \neq j$

(iii) $\nu_{i1} \geq \dots \geq \nu_{ih_i} > 0$, $h_i \geq 1$ for $i = 1, \dots, s$

(iv) $\displaystyle\sum_{i=1}^{s} \sum_{j=1}^{h_s} \nu_{ij} = n$

Furthermore, if $A_1 \neq \alpha A_0$ for some $\alpha \in \mathbb{C}\setminus\{0\}$, then also

(v) $s > 1$ or $h_1 < n$

Why are these necessary conditions?

$$\det A = \det(A_1\lambda + A_0) = \underbrace{\det A_1}_{\neq 0} \lambda^n + O(n-1)$$

$\hookrightarrow \deg \det A = n$

Regarding condition (v), assume $s = 1$, and $h_1 \geq n$.

By condition (iv), we must have $h_1 = n$ and $\nu_{11} = \dots = \nu_{1n} = 1$.

Thus, the list of elementary divisors is given by

$$\underbrace{(\lambda - \lambda_1), \dots, (\lambda - \lambda_1)}_{n \text{ times}}$$

and the invariant factors are given by

$$I_j = \lambda - \lambda_1 \quad \text{for } j = 1, \dots, n$$

In particular, $D_1 = \lambda - \lambda_1$, and therefore $\lambda - \lambda_1$ is the greatest common divisor of all (non-zero) entries of A, i.e.

$$A = A_1\lambda + A_0 = (\lambda - \lambda_1)P \quad \text{for some constant matrix } P \in \mathbb{C}^{n\times n}$$
$$= \lambda P - \lambda_1 P$$

$$\hookrightarrow \begin{cases} A_1 = P \\ A_0 = -\lambda_1 P \end{cases} \rightsquigarrow A_0 = -\lambda_1 P = -\lambda_1 A_1 \quad \lightning \text{ contradicts } A_1 \neq \alpha A_0$$

These conditions are sufficient even for symmetric matrices A_1, A_0.
(We showed that one can obtain all such Segre symbols with symmetric matrices A_1, A_0)

For this we construct matrices consisting of blocks of the form

$$\mathbb{C}[\lambda]^{\nu_{ij}\times\nu_{ij}} \ni C_1(\nu_{ij})\lambda + C_0(\lambda_i, \nu_{ij}) = \begin{pmatrix} & & & \lambda-\lambda_i \\ & & \iddots & 1 \\ & \iddots & \iddots & \\ \lambda-\lambda_i & 1 & & \end{pmatrix} \overset{\text{equiv.}}{\sim} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & (\lambda-\lambda_i)^{\nu_{ij}} \end{pmatrix}$$

$$\begin{cases} D_{\nu_{ij}} = (\lambda-\lambda_i)^{\nu_{ij}} \\ D_{\nu_{ij}-1} = 1 \\ \vdots \\ D_1 = 1 \end{cases} \rightsquigarrow \begin{cases} I_{\nu_{ij}} = (\lambda-\lambda_i)^{\nu_{ij}} \\ I_{\nu_{ij}-1} = 1 \\ \vdots \\ I_1 = 1 \end{cases}$$

The constructed matrix then looks like

$$C_1\lambda + C_0 = \begin{pmatrix} C_1(\nu_{11})\lambda + C_0(\lambda_1,\nu_{11}) & & & & \\ & \ddots & & & \\ & & C_1(\nu_{1h_1})\lambda + C_0(\lambda_1,\nu_{1h_1}) & & & \\ & & & \ddots & & \\ & & & & C_1(\nu_{s1})\lambda + C_0(\lambda_s,\nu_{s1}) & \\ & & & & & \ddots & \\ & & & & & & C_1(\nu_{sh_s})\lambda + C_0(\lambda_s,\nu_{sh_s}) \end{pmatrix} \in \mathbb{C}[\lambda]^{n\times n}$$

$$\overset{\text{equiv.}}{\sim} \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & (\lambda-\lambda_1)^{\nu_{11}} & & & & & \\ & & & & \ddots & & & & \\ & & & & & (\lambda-\lambda_1)^{\nu_{1h_1}} & & & \\ & & & & & & \ddots & & \\ & & & & & & & (\lambda-\lambda_s)^{\nu_{s1}} & \\ & & & & & & & & \ddots \\ & & & & & & & & & (\lambda-\lambda_s)^{\nu_{sh_s}} \end{pmatrix}$$

$$D_n = (\lambda-\lambda_1)^{\nu_{11}+\dots+\nu_{1h_1}} \cdot \ldots \cdot (\lambda-\lambda_s)^{\nu_{s1}+\dots+\nu_{sh_s}}$$

$$D_{n-1} = (\lambda-\lambda_1)^{\nu_{12}+\dots+\nu_{1h_1}} \cdot \ldots \cdot (\lambda-\lambda_s)^{\nu_{s2}+\dots+\nu_{sh_s}}$$

The $(n-1)\times(n-1)$ minors (non-zero) are of the form

$$\underbrace{\begin{array}{l} (\lambda-\lambda_1)^{\nu_{11}+\dots+\nu_{1h_1}} \cdot \ldots \\ (\lambda-\lambda_1)^{\nu_{12}+\dots+\nu_{1h_1}} \cdot \ldots \\ (\lambda-\lambda_1)^{\nu_{11}+\nu_{13}+\dots+\nu_{1h_1}} \cdot \ldots \end{array}}_{\text{g.c.d. is } (\lambda-\lambda_1)^{\min\{\nu_{11}+\dots+\nu_{1h_1},\ \nu_{12}+\dots+\nu_{1h_1},\ \nu_{11}+\nu_{13}+\dots+\nu_{1h_1},\ \dots\}}}$$

$$= (\lambda-\lambda_1)^{\nu_{12}+\dots+\nu_{1h_1}}$$

$$D_{n-2} = (\lambda-\lambda_1)^{\nu_{13}+\dots+\nu_{1h_1}} \cdot \ldots \cdot (\lambda-\lambda_s)^{\nu_{s3}+\dots+\nu_{sh_s}}$$

$$\vdots$$

Thus, the invariant factors are

$$I_n = \frac{D_n}{D_{n-1}} = (\lambda-\lambda_1)^{\nu_{11}} \cdot \ldots \cdot (\lambda-\lambda_s)^{\nu_{s1}}$$

$$I_{n-1} = \frac{D_{n-1}}{D_{n-2}} = (\lambda-\lambda_1)^{\nu_{12}} \cdot \ldots \cdot (\lambda-\lambda_s)^{\nu_{s2}}$$

$$\vdots$$

and therefore the elementary divisors

$$(\lambda-\lambda_1)^{\nu_{11}}, \dots, (\lambda-\lambda_1)^{\nu_{1h_1}}, \dots, (\lambda-\lambda_s)^{\nu_{s1}}, \dots, (\lambda-\lambda_s)^{\nu_{sh_s}}.$$

Example:

1) List of Segre symbols for $n=2$ (for symmetric matrices A_1, A_0, $\det A_1 \neq 0$, $A_1 \neq \alpha A_0$)

$[(\lambda_1 : 1), (\lambda_2 : 1)]$

$[(\lambda_1 : 2)]$

2) List of Segre symbols for $n=3$

$[(\lambda_1 : 1), (\lambda_2 : 1), (\lambda_3 : 1)]$

$[(\lambda_1 : 2), (\lambda_2 : 1)]$

$[(\lambda_1 : 1\,1), (\lambda_2 : 1)]$

$[(\lambda_1 : 3)]$

$[(\lambda_1 : 2\,1)]$

$\det(A_1 \lambda + A_0) = 0$