Stress Matrices and M Matrices

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What is an M Matrix?

Given a finite graph $G=(V,E)$, it is an n-by-n symmetric matrix $M = (m_{ij})$, $|V| = n$, such that:

(M1) $m_{ij} < 0$ for ij in E, $m_{ij} = 0$ for ij not in E.

(M2) M has exactly one negative eigenvalue, counting multiplicity.

(M3) If $X=(x_{ij})$ is an n-by-n matrix such that $MX=0$, and $x_{ij}=0$, when $i=j$ or ij is in E, then $X=0$.

Lovasz's Theorem

Let G=(V,E) be the one skeleton of a convex 3-dimensional polytope. Then there is an M matrix for G such that M has exactly three 0 eigenvalues, as well as exactly one negative eigenvalue.

What is a Stress Matrix?

Given a finite graph $G=(V,E)$, it is an n-by-n symmetric matrix $\boldsymbol{\Omega} = (-\omega_{ij})$, $|V| = n$, such that:

(S1) $\omega_{ij} = 0$ when ij is not in E, and i is not j.

(S2) $[1,1, \ldots, 1]\, \boldsymbol{\Omega} = 0$, which determines the diagonal elements.

Properties of a Stress Matrix

- If $P\Omega = 0$, where $P=[p_1, p_2, \ldots, p_n]$, we say the configuration $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is in *equilibrium* wrt Ω.
- The dimension of the affine span of **p** is at most n - 1 - rank(Ω).
- If the dimension of the affine span of **p** is n - 1 - rank(Ω) and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ is in equilibrium wrt Ω, then q is an affine image of **p**.

(In this case we say **p** is *universal wrt* Ω.)

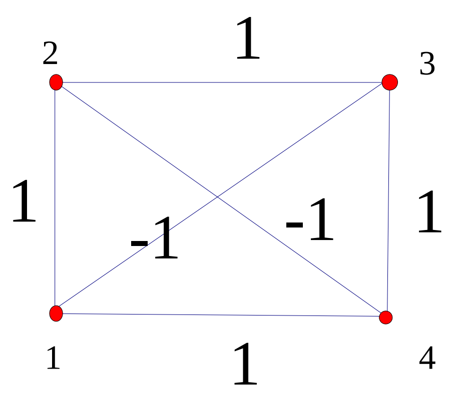
Conics at Infinity

If v_1, v_2, … are vectors in E^d, we say that they *lie on a conic at infinity* if there is a non-zero symmetric matrix C such that $v_i^T C v_i = 0$, where $()^T$ is the transpose.

Suppose that the edges of a graph G are labeled either a cable or a strut. We say a configuration **q**, whose vertices are the vertices of G, is *dominated by* **p** if the cables of **q** are not increased, and struts are not decreased. We call G(**p**) a *tensegrity*.

Global Rigidity

If every configuration that is dominated by **p** is congruent **p**, we say G(**p**) is *globally rigid*.



$$\Omega = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

Main Result

Theorem: If a configuration **p** has an equilibrium stress **ω**, with $\omega_{ij} > 0$ for cables, $\omega_{ij} < 0$ for struts, (called a *proper stress* for G) such that

i.) The member directions $p_i - p_j$, for ij in E, do not lie on a conic at infinity,

ii.) Ω is positive semidefinite, and

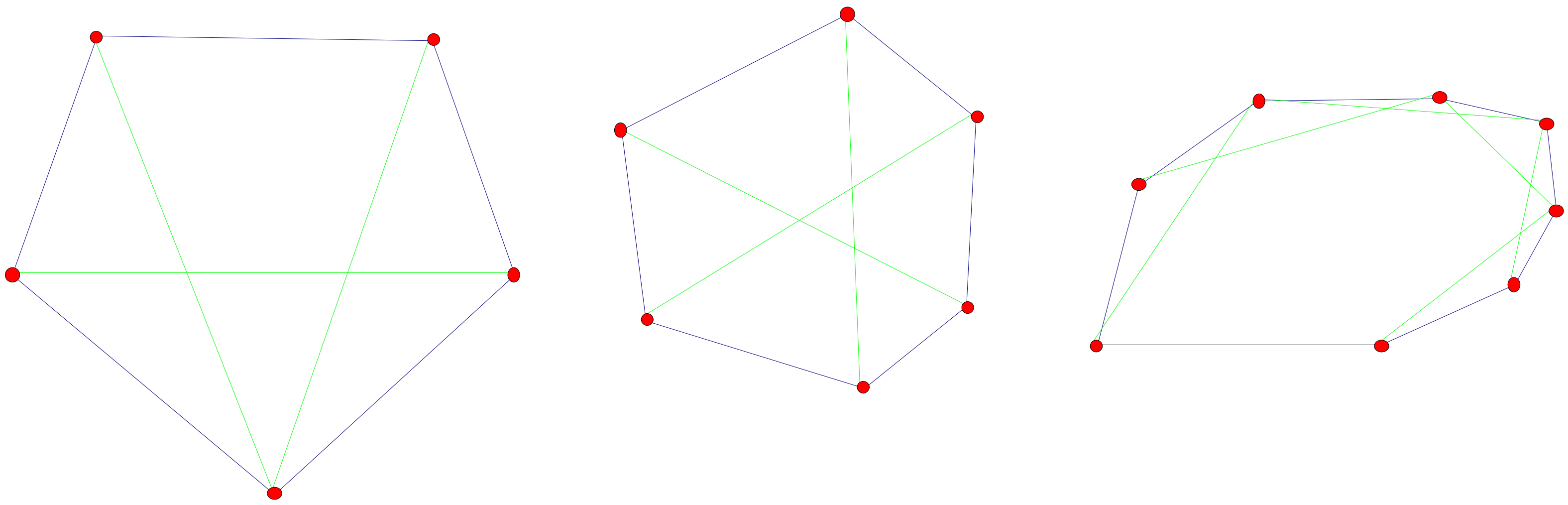
iii.) **p** is universal with respect to **ω**,

then G(**p**) is globally rigid.

Any configuration that satisfies the hypothesis above is called *superstable*.

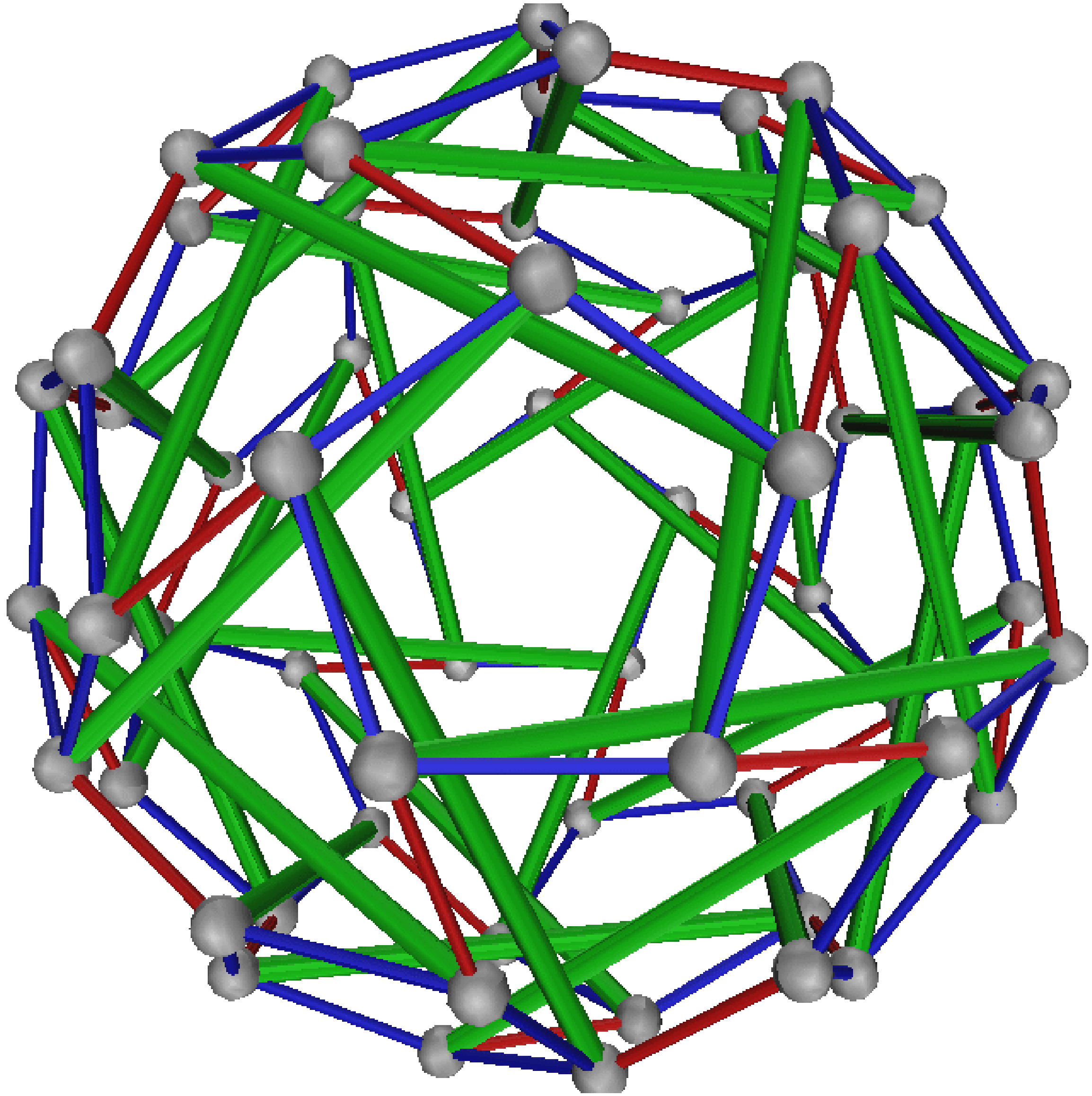
Examples

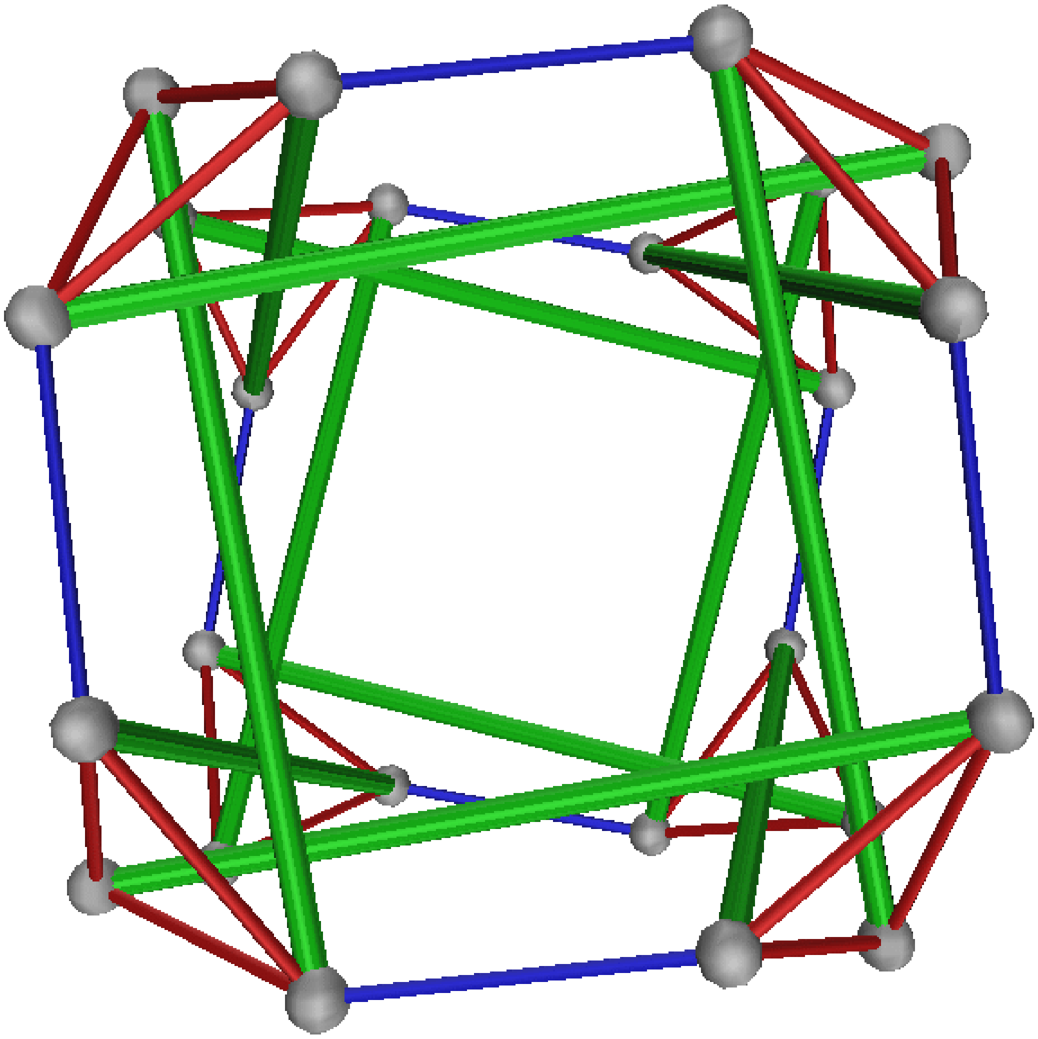
Some superstable polygons in the plane. Any convex polygon in the plane, with cables on the outside, struts inside, and a proper stress is superstable.

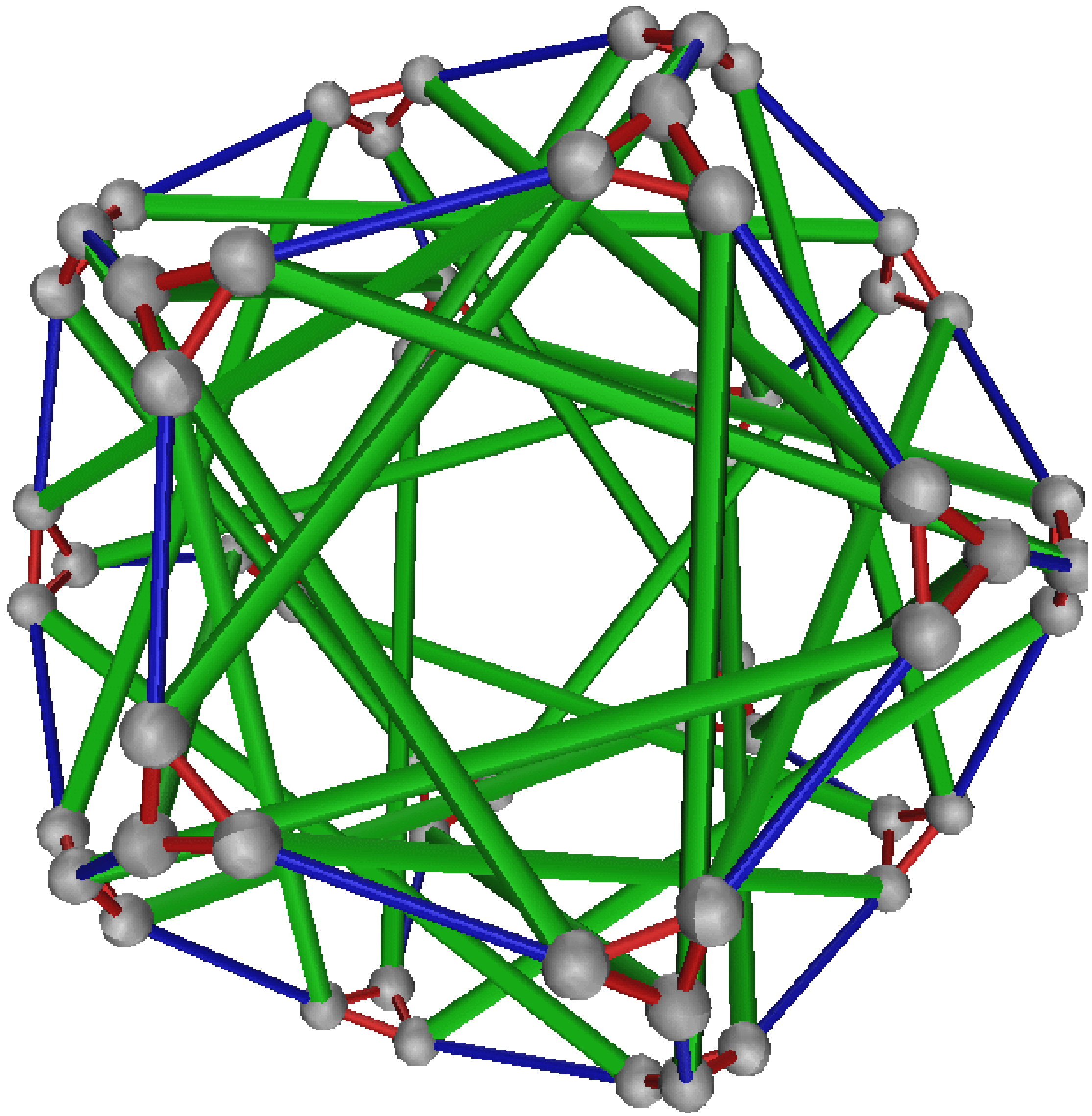


Conjecture

If G(**p**) is a tensegrity obtained from a convex 3-dimensional polytope K whose edges form the cables of G, struts are some set of interior diagonals, and such that G(**p**) has a proper equilibrium stress. Then G(**p**) is super stable.

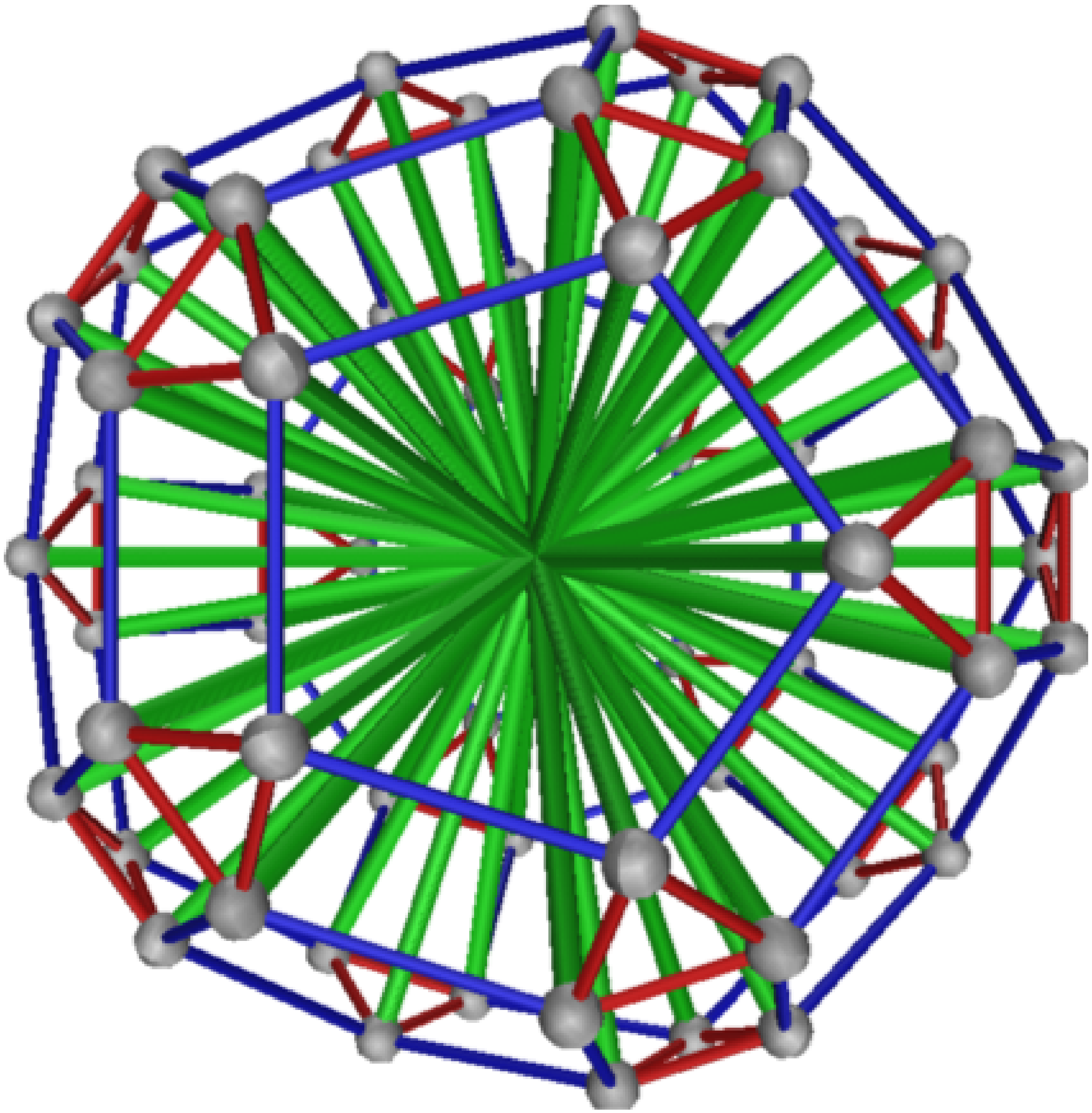






Karoly Bezdek's Conjecture

If K in the previous conjecture is centrally symmetric and the struts connect antipodal vertices, then G(**p**) is super stable.



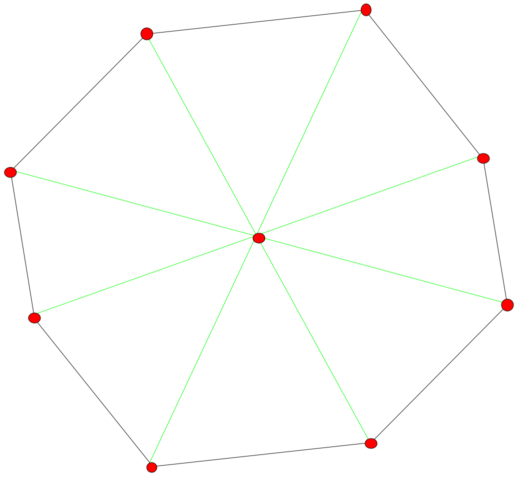
M -> Ω

M matrices correspond naturally to stress matrices. Add one new row and column to M to get a matrix Ω in such a way that row and column sums of Ω are 0, and this correspondence works the other direction if the stress matrix has one vertex that is connected to all the others.

Proof of Bezdek's Conjecture

Add a vertex to the center of a centrally symmetric polytope K and connect it by a strut to all the vertices of K. Using cables for edges of K gives a tensegrity G(**p**). It is possible (Maxwell-Cremona argument) to show that G(**p**) has a proper non-zero equilibrium stress. Then we get the corresponding M matrix M(K). Lovasz's Theorem implies M(K) has exactly one negative eigenvalue. Thus Ω has at most one negative eigenvalue. So it has one negative eigenvalue.

Proof continured



Think of Ω as a quadratic form. Add little stresses that are positive semi-definite to the above Ω that cancel the stress on antipodal pairs of struts. This replaces each pair of antipodal struts with a long strut connecting antipodal vertex pairs.

Proof continued

The resulting stress matrix now has at most one negative eigenvalue and one extra eigenvalue of 0. That must have come from the negative eigenvalue due to the floating vertex. So the resulting stress matrix is positive semi-definite with only three 0 eigenvalues.