Some Applications of the cosine law

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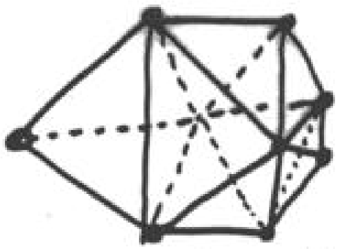
Oberwolfach

Smooth metric g_{ij}. Rijke

polyhedral surface 

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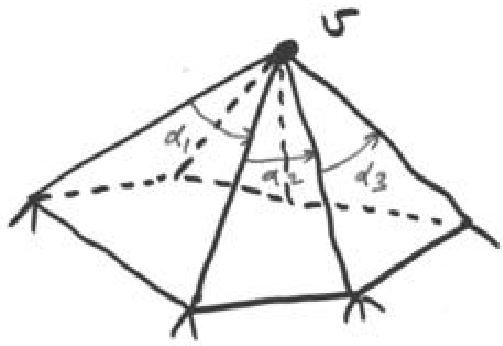
isometric gluing of triangles in $\mathbb{E}^2, \mathbb{H}^2, \ldots$

P.L. metric:

$$l : E = \{\text{all edges}\} \longrightarrow \mathbb{R}^1$$

discrete curvature:

$$K : \{\text{vertices}\} \longrightarrow \mathbb{R}$$



$$K(v) = 2\pi - \sum_i \alpha_i$$

<u>Gauss-Bonnet</u> $\quad \sum_v K(v) = 2\pi\chi(S)$

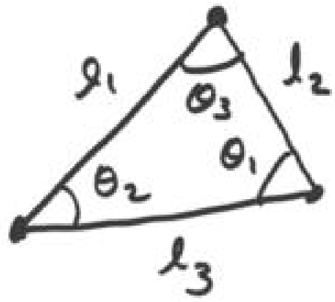
(in Euclidean triangles.)

In the smooth case: $g_{ij} \rightsquigarrow R_{ijkl}$

polyhedral surfaces: edge lengths $\rightsquigarrow$ angles

governed by the $\boxed{\text{cosine Law}}$

An Example $\mathbb{E}^2$



$$\boxed{\cos\theta_i = \frac{l_j^2 + l_k^2 - l_i^2}{2 l_j l_k}}$$

Convention $i \neq j \neq k \neq i$

consider $\theta_i = \theta_i(l_1, l_2, l_3)$

Fact: $$\boxed{\frac{1}{\sin\theta_i}\frac{\partial\theta_i}{\partial l_j} = \frac{1}{\sin\theta_j}\frac{\partial\theta_j}{\partial l_i}}$$

Derivative Cosine Law

Thus $\omega = \sum \ln\tan\left(\frac{\theta_i}{2}\right) dl_i$ **closed** 1-form

$$F(l) = \int_{(1,1,1)}^{l} \omega$$

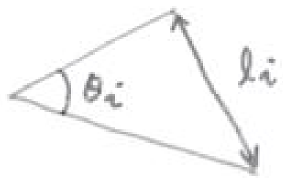
$E(2) = \{(l_1, l_2, l_3) \mid l_i + l_j > l_k\}$

space of all $\mathbb{E}^2$ triangles

Then $F = \int^l \omega : E(2) \longrightarrow \mathbb{R}$

satisfies:

(1) $\boxed{\dfrac{\partial F}{\partial l_i} = \ln \tan\left(\dfrac{\theta_i}{2}\right)}$



(2) $\left[\dfrac{\partial^2 F}{\partial l_r \partial l_s}\right] = \left[\dfrac{1}{\sin\theta_r}\dfrac{\partial \theta_r}{\partial l_s}\right]_{3\times 3}$ semi-pos. definite

F is convex.

For a polyhedral surface (Σ, T, ℓ),

triangulation edge length

define its "energy" $E(\ell)$ to be

the sum of "energies" of its triangles

$$E(\ell) = \sum_{\Delta ijk \in T} F(\ell_i, \ell_j, \ell_k)$$

<u>Then</u>

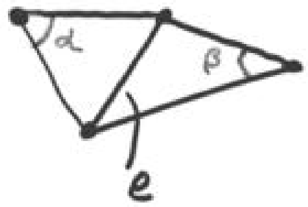
(1) $E(\ell)$ is convex

(2) $$\frac{\partial E}{\partial \ell_i} = \ln \tan\left(\frac{\alpha}{2}\right) \tan\left(\frac{\beta}{2}\right)$$

Let $\psi : \{\text{all edges}\} \longrightarrow \mathbb{R}$

$$\psi(e) = \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)$$

Edge invariant of polyhedral surface



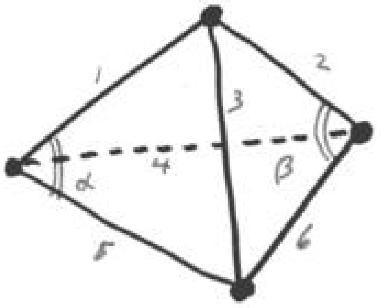
Then $\nabla E = (\ln\psi(e_1), \cdots, \ln\psi(e_n))$

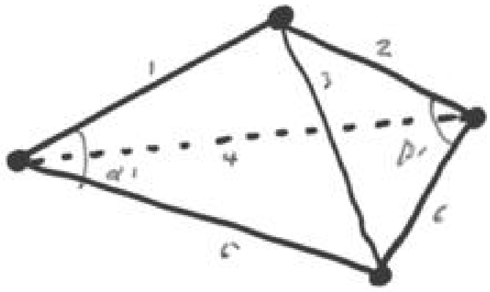
<u>Well known</u>: $W : \Omega \to \mathbb{R}^1$ strictly convex, then $x \mapsto \nabla W$ is 1-1.

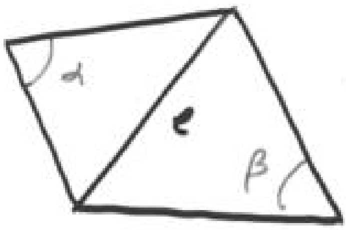
$\Rightarrow$

<u>Thm 1.</u> A polyhedral surface is determined up to isometry and scaling by its edge invariant. $\psi : \{\text{all edges}\} \longrightarrow \mathbb{R}$.

(in $\mathbb{E}^2$ triangles)







$$\psi(e) = \tan\left(\frac{\alpha}{2}\right) \tan\left(\frac{\beta}{2}\right)$$

$$\alpha + \beta < \pi \iff \psi(e) < 1.$$

Q1. Can you find all functions

$F(x_1, x_2, x_3)$ defined on the space of triangles

$(\mathbb{E}^2, \mathbb{H}^2, \mathbb{S}^2)$ so that

$$\frac{\partial F}{\partial x_i} = h(y_i)$$

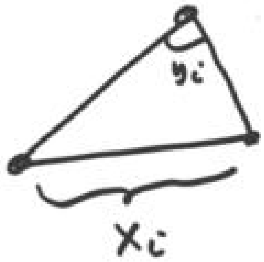
or even
$$\frac{\partial F}{\partial g(x_i)} = h(y_i) \qquad ?$$

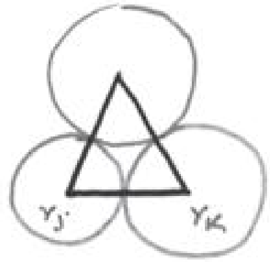
$\Longleftrightarrow$ Can you find all closed 1-forms

$$\sum_i h(y_i)\, dx_i \ ?$$

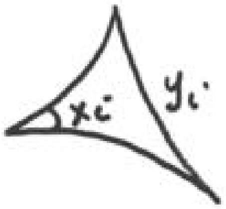
$$\sum_i h(y_i)\, d\, g(x_i) \ ?$$

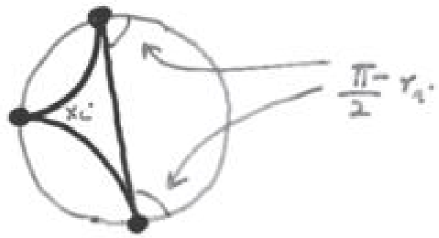
Parametrizations of triangles





$x_i = r_j + r_k$





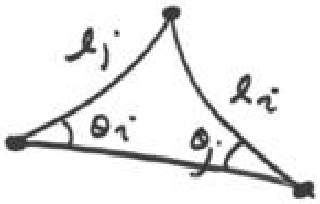
$$r_i = \frac{1}{2}(x_j + x_k - x_i)$$

is an interesting parameter of $\triangle$

Q2. Similar to Q1:

Functions $F(\,,)$ $\quad \dfrac{\partial F}{\partial g(r_i)} = h(y_i)$?

The cosine laws



$\mathbb{E}^2$ or $\mathbb{H}^2$ or S^2

(S^2)
$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j \cdot \cos\theta_k}{\sin\theta_j \, \sin\theta_k}$$

$(\mathbb{H}^2)$
$$\cosh l_i = \frac{\cos\theta_i + \cos\theta_j \cdot \cos\theta_k}{\sin\theta_j \cdot \sin\theta_k}$$

$(\mathbb{E}^2)$
$$1 = \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \cdot \sin\theta_k}$$

There are 3 more expressing θ_i in terms of l_i, l_2, l_3

<u>convention</u> $i \neq j \neq k \neq i$

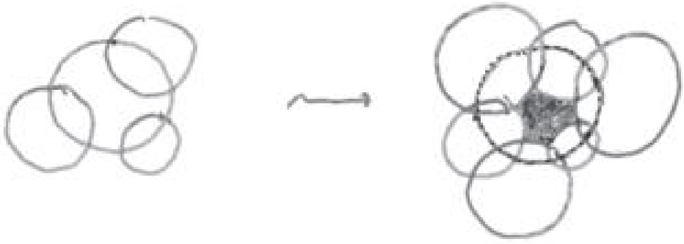
In fact, there is only one formula:

$$\cos(\sqrt{\lambda}\, l_i) = \frac{\cos\theta_i + \cos\theta_j \cdot \cos\theta_k}{\sin\theta_j \cdot \sin\theta_k}$$

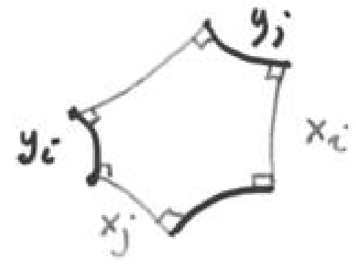
$\lambda = -1, 0, 1$ is the curvature of

$\mathbb{H}^2, \mathbb{E}^2, S^2$

Another cosine law coming from three pairwise disjoint circles:



Hyperbolic right-angled hexagon



$$\boxed{\cosh y_i = \frac{\cosh x_i + \cosh x_j \cosh x_k}{\sinh x_j \sinh x_k}}$$

We should consider cosine law:

$$y = (y_1, y_2, y_3) \in \mathbb{C}^3$$

$$x = (x_1, x_2, x_3) \in \mathbb{C}^3$$

$$y = y(x) : \Omega \longrightarrow \mathbb{C}^3$$

Ω open $\mathbb{C}^3$

$$\boxed{\cos y_i = \frac{\cos x_i + \cos x_j \cos x_k}{\sin x_j \sin x_k}}$$

$i \neq j \neq k \neq i$

Let $A_{ijk} = \sin y_i \sin x_j \sin x_k$

$\boxed{\text{Thm 2}}$

(a) $A_{ijk} = A_{jki} = A$ (sine law)

(b) $$\frac{\partial y_i}{\partial x_i} = \frac{\sin x_i}{A} = \frac{\sin y_i}{\tilde{A}}$$

(c) $$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_i}{\partial x_i} \cos y_k$$

(d) $$\cos x_i = \frac{\cos y_i - \cos y_j \cdot \cos y_k}{\sin y_j \cdot \sin y_k} \qquad (A \neq 0)$$

<u>RM</u>. 1. We call these (b),(c) the derivative cosine laws.

Kind of "Bianchi identities" ?

2. (d) is the duality of edge lengths and angles.

Corollary. $\dfrac{\partial y_i}{\partial x_j} \Big/ \dfrac{\partial y_j}{\partial x_i} = \dfrac{\sin y_i}{\sin y_j}$.

Suppose $\omega = \sum' f(y_i)\, dx_i$ is closed

$$\Leftrightarrow \quad \frac{\partial f(y_i)}{\partial x_j} = \frac{\partial f(y_j)}{\partial x_i}$$

$$\Leftrightarrow \quad f'(y_i)\frac{\partial y_i}{\partial x_j} = f'(y_j)\frac{\partial y_j}{\partial x_i}$$

corollary

$$\Leftrightarrow \quad f'(y_i)\sin y_i = f'(y_j)\sin y_j$$

$$\Rightarrow \quad f'(y_i)\sin y_i = c$$

$$\Rightarrow \quad f'(t) = \frac{c}{\sin t}$$

$$\Rightarrow \quad f(t) = c \ln\tan\left(\frac{t}{2}\right) + c'$$

$$\omega_i = \sum \ln\tan\left(\frac{y_i}{2}\right) dx_i$$

Thm 3 Up to scaling, all closed 1-forms

$$\sum_i f(y_i)\, d\, g(x_i) \quad \text{are}$$

$$\omega_N = \sum_{i=1}^{3} \left(\int^{y_i} \sin^{-N}(t)\, dt \right) d\left(\int^{x_i} \sin^{N-1}(t)\, dt \right)$$

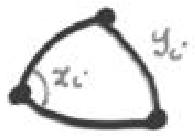
$$= \sum_{i=1}^{3} \left(\int^{y_i} \sin^{-N}(t)\, dt \right) \sin^{N-1}(x_i)\, dx_i$$

$(N \in \mathbb{Z})$

$$\omega_0 = \sum_i \frac{y_i}{\sin x_i}\, dx_i$$

$$\omega_1 = \sum_i \ln \tan\left(\frac{y_i}{2}\right) dx_i$$

Furthermore, for spherical triangles:



$\int \omega_N$ is strictly convex in (u_1, u_2, u_3)

where $u_i = \int^{x_i} \sin^{N-1}(t)\, dt$

$N \in \mathbb{R}$

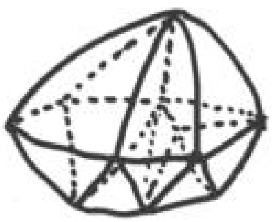
Each of $\int \omega_N$, $N \in \mathbb{R}$, produces a result similar to thm 1.

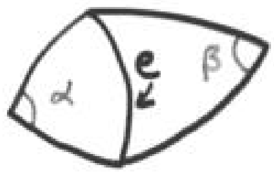
$$\int \sum_i y_i \, d\left(\ln \tan \frac{x_i}{2}\right) \Rightarrow$$

<u>Thm 4</u>. A spherical polytope surface P is determined up to isometry by its edge invariant

$$\Phi : \{\text{all edges}\} \longrightarrow \mathbb{R}$$

$$\Phi(e) = \alpha + \beta$$





Furthermore, given $\Phi : \{\text{edges}\} \to [0, \pi)$

there is a spherical polyhedral surface P

with edge invariant Φ iff:

for any subset $X \subset \{\text{triangles}\}$

$$\pi |X| < \sum_{e \in E(X)} \Phi(e)$$

where $E(x)$ is the set of all edges in X.

(proved by Ren Guo + L.)

Legendre Transform (LT)

$\mathcal{U}, V$ open in $\mathbb{R}^n$ (connected, $H_1 = 0$)

$t = t(s) : \mathcal{U} \longrightarrow V$ diffeo. s.t.

$$\frac{\partial t_i}{\partial s_j} = \frac{\partial t_j}{\partial s_i}$$

Then we say

$$f(s) = \int^s \sum_i t_i \, ds_i \xleftrightarrow{LT} g(t) = \int^t \sum_i s_i \, dt_i$$

In particular,

$$\int \omega_N \xleftrightarrow{LT} \int \omega_{-N+1}$$

$$\int \omega_0 \xleftrightarrow{LT} \int \omega_1$$

$$\int^N \omega_1 = \int^N \sum_i \ln \tan\left(\frac{x_i}{2}\right) dx_i = \text{comb. dilogarithms}$$

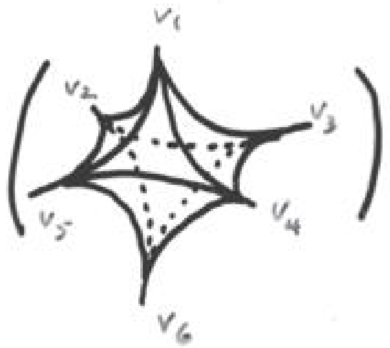
I don't know the geometric meaning of $\int \omega_N$

except $\int \omega_1$.

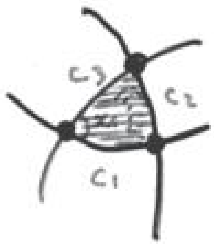
For a spherical triangle 

,

$$-4\int_{(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})}^{x} \sum_i \ln\tan\left(\frac{y_i}{2}\right) dx_i + 16\,\Lambda\left(\frac{\pi}{4}\right)$$

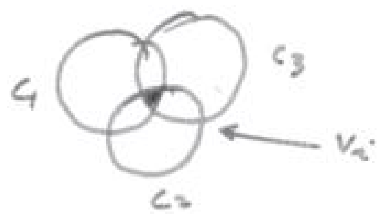
= hyperbolic volume (

)



$$\{v_1, v_2, \dots, v_6\} = \bigcup (c_i \cap c_j)$$

$$\subset S^2_\infty \subset \mathbb{H}^3$$



closed 1-forms $\sum h(y_i)\, d\, g(r_i)$, $\sum g(r_i)\, d\, h(y_i)$

They exist due to the remarkable work of Colin de Verdiere, Leibon.

Colin de Verdiere '91. For a hyperbolic 

$$\sum_i \frac{y_i}{\sinh(r_i)}\, dr_i \quad \text{is closed and}$$

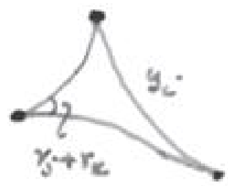
$$\int \sum_i \frac{y_i}{\sinh(r_i)}\, dr_i \quad \text{strictly convex in } (u_1, u_2, u_3)$$

where $u_i = \ln \tanh(r_i/2)$

$\Rightarrow$ a variational principle for circle packing

$\Rightarrow$ rigidity theorem of circle packing $(\mathbb{E}^2, S^2)$

Bobenko - Springborn

G. Leibon '02. For a hyperbolic 

$$\sum_i \ln \sinh\left(\frac{y_i}{2}\right) dr_i \quad \text{is closed and its}$$

integration is strictly concave.

<u>Thm 5</u>. Up to scaling, all closed 1-forms $\sum f(y_i)\, dg(r_i)$

are

$$\eta_N = \sum_i \int^{y_i} \cot^{N+1}\left(\frac{t}{2}\right) dt \; d\left(\int^{r_i} \cos^N(t)\, dt \right)$$

$$= \sum_i \left(\int^{y_i} \cot^{N+1}\left(\frac{t}{2}\right) dt \right) \cos^N(r_i)\, dr_i$$

$(N \in \mathbb{Z})$

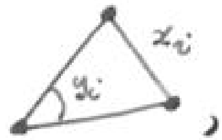
$$\eta_{-1} = \sum_i \frac{y_i}{\cos(r_i)}\, dr_i$$

(Colin de Verdière)

$$\eta_0 = 2 \sum_i \ln \sin\left(\frac{y_i}{2}\right) dr_i$$

(Leibon)

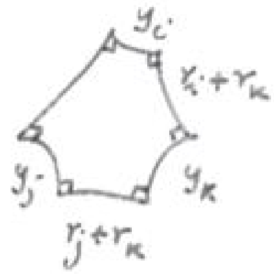
<u>Thm 6</u>

(a) For a Euclidean triangle 

,

$$\sum_{i=1}^{3} \frac{\int^{y_i} \sin^N t \, dt}{x_i^{N+1}} \, dx_i \qquad (N \in \mathbb{R})$$

is closed. Its integration is convex. in $u_i = \int^{x_i} \frac{1}{t^{N+1}} dt$.

(b) For a hyperbolic r.a. hexagon 

$$\sum_i \ln \cosh\left(\frac{y_i}{2}\right) dr_i$$

is closed. Its integration is strictly concave.

By L.T. $\Rightarrow$ $\displaystyle\int \sum_i r_i \, d\left(\ln \cosh\left(\frac{y_i}{2}\right)\right)$

strictly convex.

An application to Teichmüller theory

$\Sigma_{g,r}$ = genus g, $r>0$ boundary comp.

$$\chi(\Sigma)<0 \iff 2g+r>2$$

<u>Uniformization Thm</u> (Poincaré Koebe)

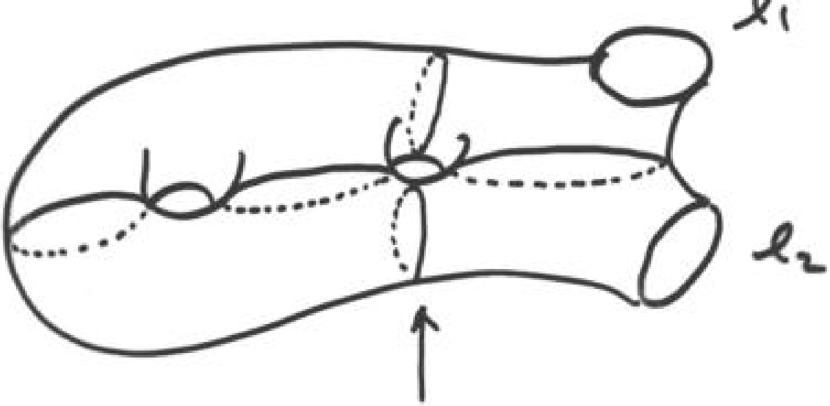
Given any Riemannian metric g on Σ there exists a unique hyperbolic metric d on Σ conformal to g s.t $\partial\Sigma$ consists of geodesics.

$T(\Sigma_{g,r}, \ell_1, \cdots, \ell_r) = \{\text{all such metrics}\}/\sim$

$\sim$: isometric by an isometry $\simeq$ id

<u>Known</u>: $T_{g,r}(\ell_1, \cdots, \ell_r) \cong \mathbb{R}^{6g-6+2r}$

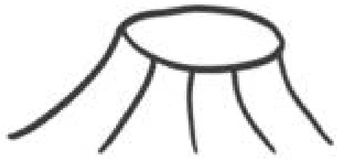
A coordinate : Fenchel-Nielsen



(length, twisting) $\in \mathbb{R}_{>0} \times \mathbb{R}$

polyhedral surfaces

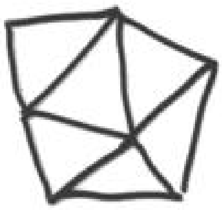
Cone metrics | hy. metric geodesic boundary



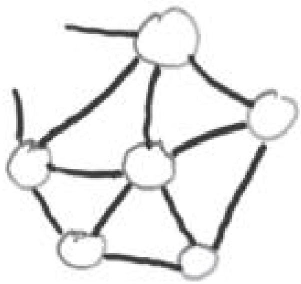
cone angle $\longleftrightarrow$ length of boundary

Cone metrics w/ pres. curvature $\longleftrightarrow$ $T_{g,r}(\ell_1, \dots, \ell_r)$

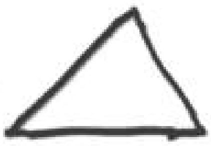
Triangulation $\Sigma_{g,0}$ | Ideal Triangulation $\Sigma_{g,r}$



Truncation $\longrightarrow$



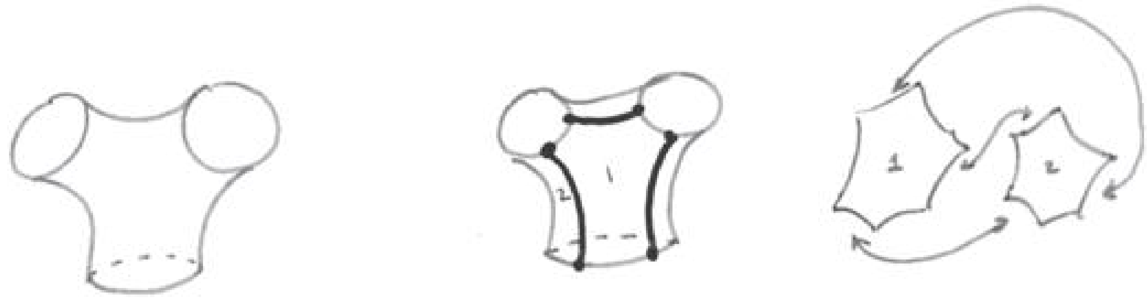
triangle $\longrightarrow$ hexagon



<u>Def</u> A hexagonal decomposition of $\Sigma_{g,r}$

: union of hexagons with vertices in $\partial\Sigma_{g,r}$

of valency = 3.



Fix a hexagonal decomposition of $\Sigma_{g,r}$

$\Leftrightarrow$ a fixed way of identifying pairs of

edges of some hexagons by homeomorphisms.

$[d] \in T(\Sigma_{g,r})$

$\Leftrightarrow$ the same way of identifying pairs of

edges of hyperbolic r.a. hexagons by isometries.

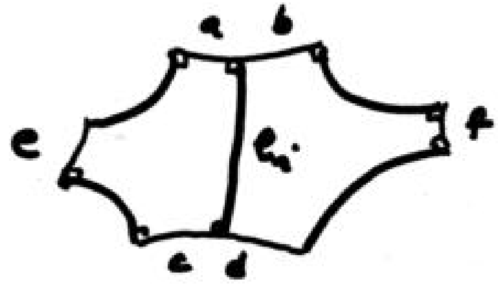
Let $e_1, \cdots, e_N$, $N = 6g-6+3r$ be

the edges in $\mathrm{int}(\Sigma_{g,r})$

Let $[d] \in T(\Sigma_{g,r})$.

For e_i, define

$$\chi(e_i) = x_i = a+b+c+d-e-f$$



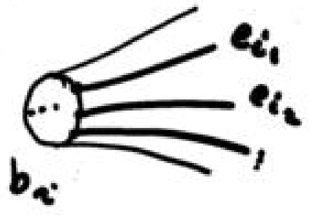
Thm 8. The map

$$x : T(\Sigma_{g,r}, l_1, \cdots, l_r) \longrightarrow \mathbb{R}^{6g-6+3r}$$

is a smooth embedding whose image is the convex polytope

$$\{(x_1, \cdots, x_N) \mid (1), (2)\}$$

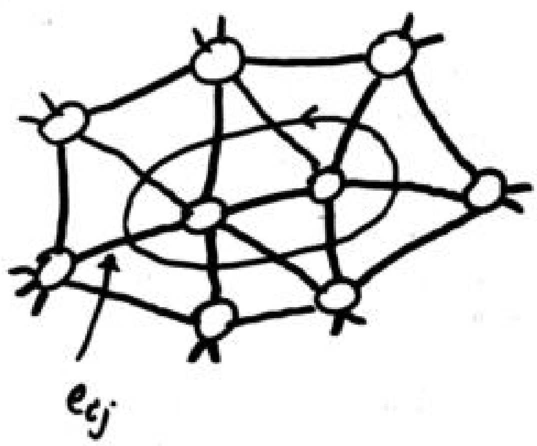
(1): $$\sum_j x(e_{i_j}) = l_i$$



(prescribing curvature)

(2) $$\sum_j x(e_{i_j}) > 0$$

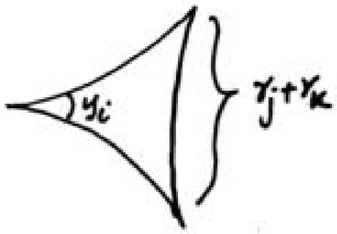
along any edge cycle



Summary

It is a good idea to take derivatives of classical geometric identities.

Thank you

For hyperbolic triangle 

the 1-forms

$$\sum_i \left(\int^{y_i} \tan^{N+1}\left(\frac{t}{2}\right) dt \right) \sinh^N(r_i)\, dr_i$$

are closed for $N \in \mathbb{R}$.

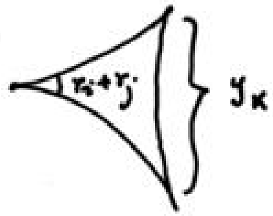
Its integration is strictly convex in

(u_1, u_2, u_3) where $u_i = \int^{r_i} \sinh^N(t)\, dt$

For $N = -1$,

$$\sum_i \frac{y_i}{\sinh(r_i)}\, dr_i \qquad \text{(Colin de Verdiere)}$$

Ethan

For a hyperbolic triangle 

the closed 1-forms

$$\sum_i \left(\int^{y_i} \coth^{N+1}\left(\frac{t}{2}\right) dt \right) \cos^N r_i \, dr_i$$

($N \in \mathbb{R}$)

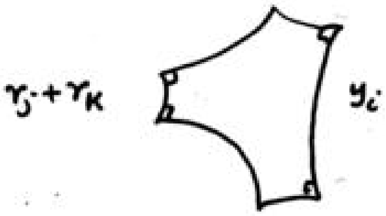
have integrations which are

strictly concave in $u_i = \int^{r_i} \cos^N t \, dt$.

$N=0$ $\quad \sum_i \ln \sinh\left(\frac{y_i}{2}\right) dr_i$ $\quad$ (Leibon)

Escher

For hyperbolic right-angled hexagon



$$\sum_i \left(\int^{y_i} \tanh^{N+1}(t)\, dt \right) \cosh^N(r_i)\, dr_i$$

is closed. Its integration is

strictly concave in $u_i = \int^{r_i} \cosh^N(t)\, dt$

$N = 0$ $$\sum_i \ln \coth\left(\frac{y_i}{2}\right) dr_i$$

cosh

[illegible]