# Discrete Conformal Maps

## Linear and Non-Linear

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**Montpellier II**

Oberwolfach Discrete Differential Geometry '06

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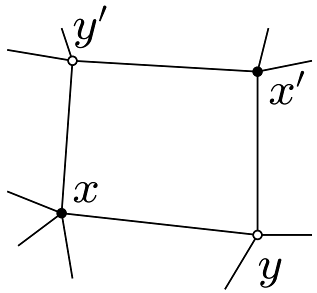
$$f \text{ is holomorphic } \Rightarrow f(z) = \begin{cases} az + b + o(z), \\ \\ \dfrac{az+b}{cz+d} + o(z^2). \end{cases}$$

Leads to two different notions of discrete conformality on a quad-graph :

– Preserving diagonal ratio (linear),

– or preserving cross-ratio (Moebius invariant).




Diagonals ratio :

$$\frac{y'-y}{x'-x} = i\,\rho$$

Crossratio :

$$\frac{y'-x'}{x'-y}\,\frac{y-x}{x-y'} = q$$

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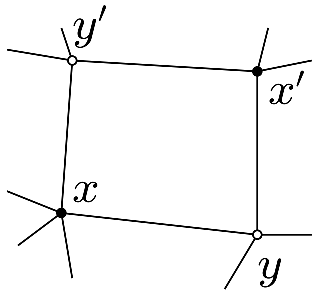


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$$\frac{y'-y}{x'-x} = i\,\rho = \frac{f(y')-f(y)}{f(x')-f(x)}$$

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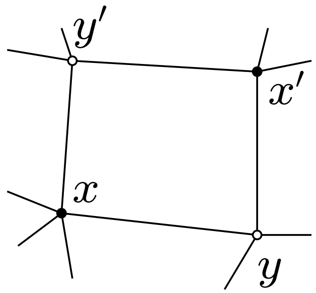


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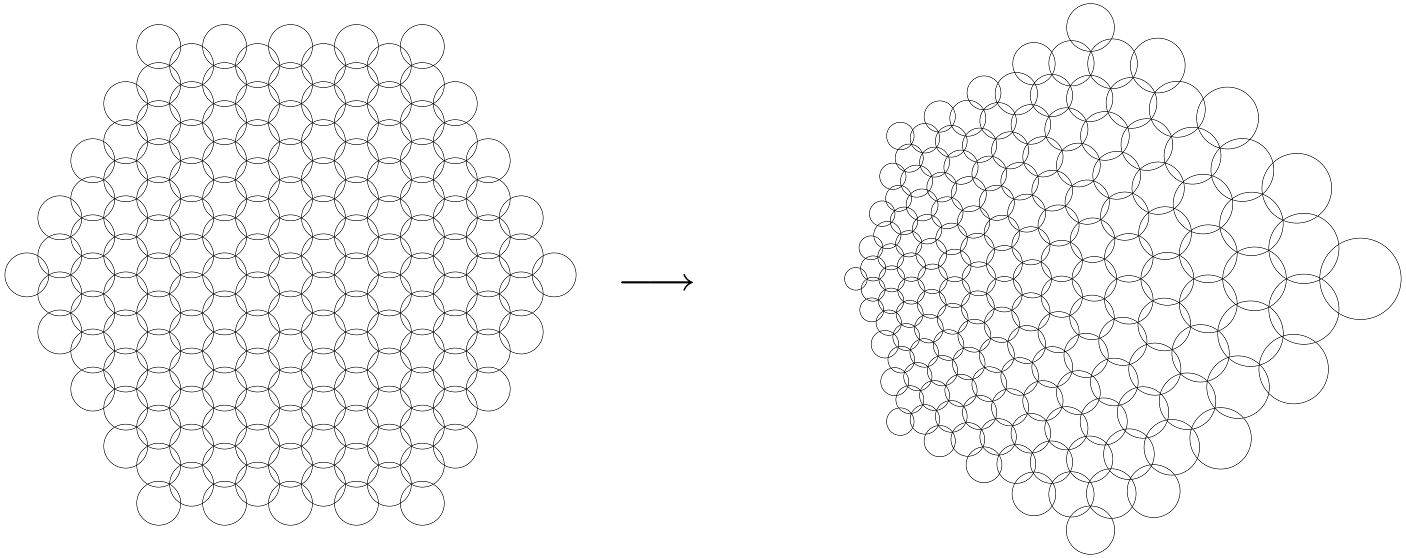
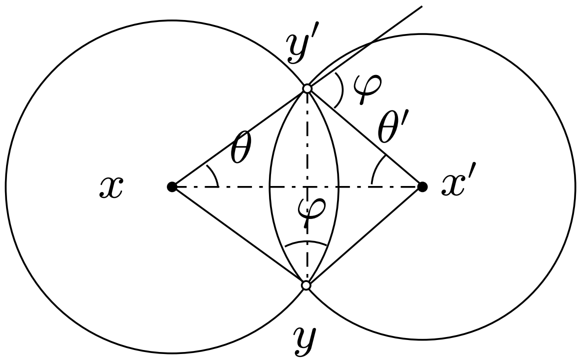
$$= \frac{f(y')-f(x')}{f(x')-f(y)}\,\frac{f(y)-f(x)}{f(x)-f(y')}$$

Circles patterns are a particular case :






$$q = e^{-2(\theta+\theta')}$$
$$= e^{-2\varphi}$$

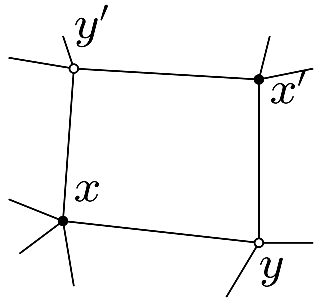
$$\rho = \frac{\cos(\theta-\theta')-\cos(\varphi)}{\sin(\varphi)}$$

# Hirota system

For every **cross-ratio preserving** function $F$, there exists a function $f$ with

$$F(y) - F(x) = f(x)\, f(y)\, (y - x) = \text{“}F'(z)\, dz\text{”}.$$

This imposes on a face $(x, y, x', y')$,




$$f(x)\, f(y)\, (y - x) + f(y)\, f(x')\, (x' - y) + f(x')\, f(y')\, (y' - x') + f(y')\, f(x)\, (x - y') = 0$$

to be understood as a Morera equation $\oint F'(z)\, dz = 0$.

Circles pattern case : $f$ is real at the centers and unitary at the intersections :

$$F(y) - F(x) = r(x)\, e^{i\,\theta(y)}\, (y - x).$$

Both definitions can be understood as a Morera equation $\oint F'(z)\,dz = 0.$ with discrete integration of functions whether

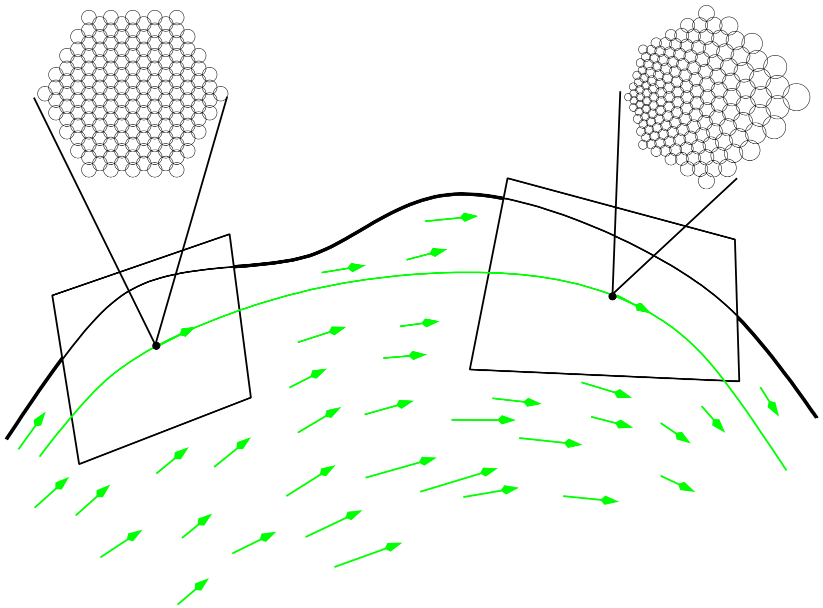
– by the *geometric* mean for the **diagonals ratio preserving** maps

$$\int_{(x,y)} g\,dZ := \sqrt{g(x)\,g(y)}\,(y-x).$$

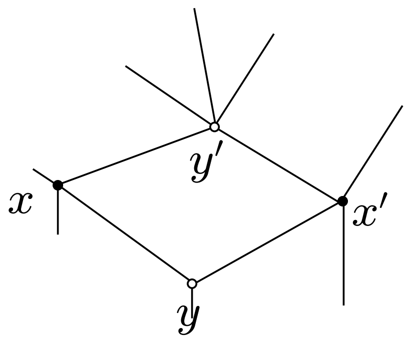
– or by the *arithmetic* mean for the **diagonals ratio preserving** maps

$$\int_{(x,y)} g\,dZ := \frac{g(x)+g(y)}{2}\,(y-x).$$

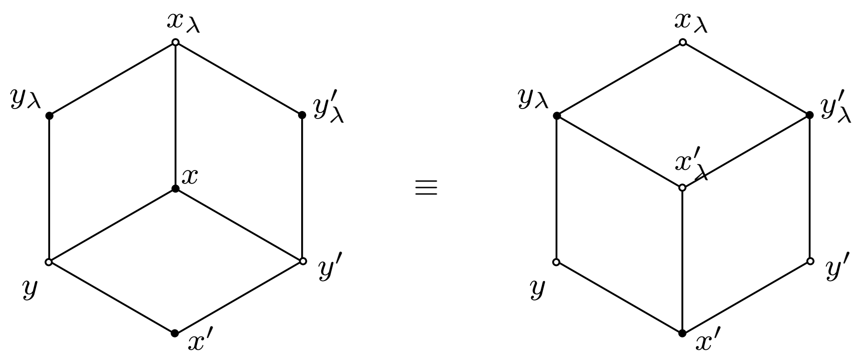
The linear case is a linearization of the quadratic one :



# Bäcklund transformation




We impose "vertically" the same equation as "horizontally".
The $3D$ consistency or **Yang-Baxter** equation yields integrability for rhombic quad-graphs :



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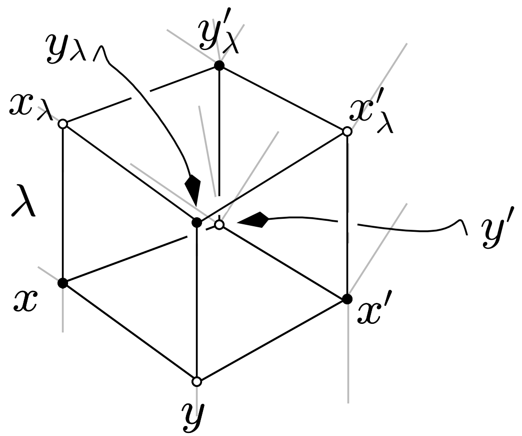


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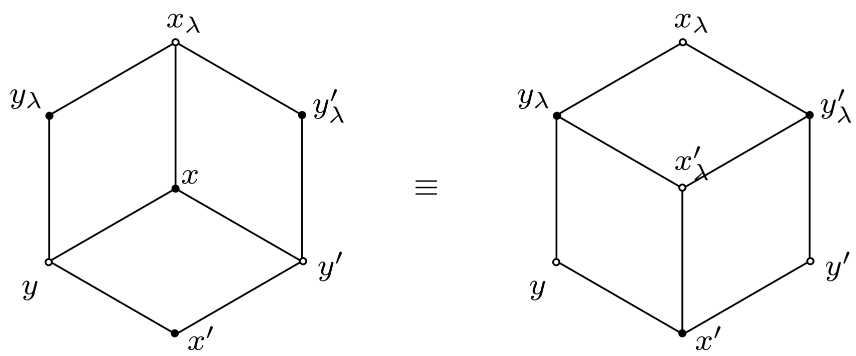

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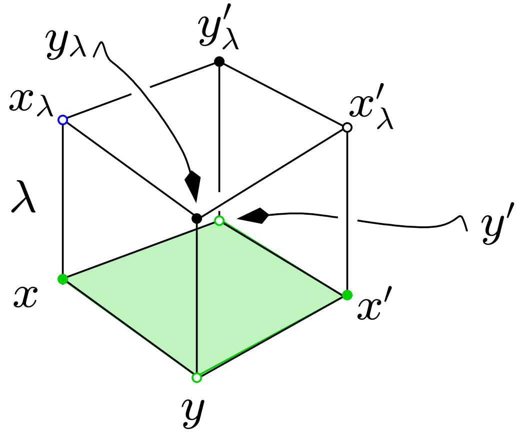
$$i\,\rho = \frac{y-y'}{x-x'}$$

$$q = \frac{(y-x)^2}{(y'-x')^2}$$

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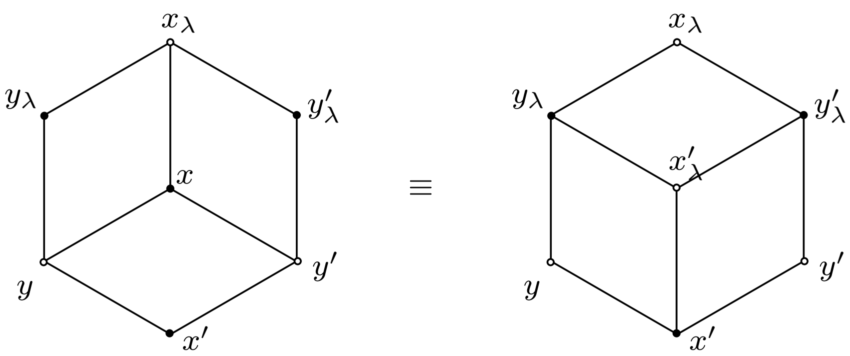
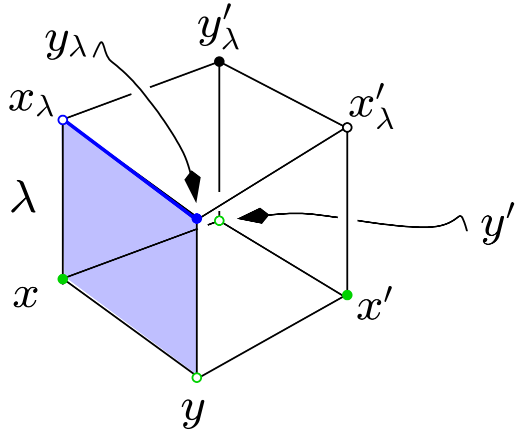


# Bäcklund transformation

$$i\,\rho = \frac{y-x+\lambda}{x-y+\lambda}$$

$$q = \frac{(y-x)^2}{\lambda^2}$$

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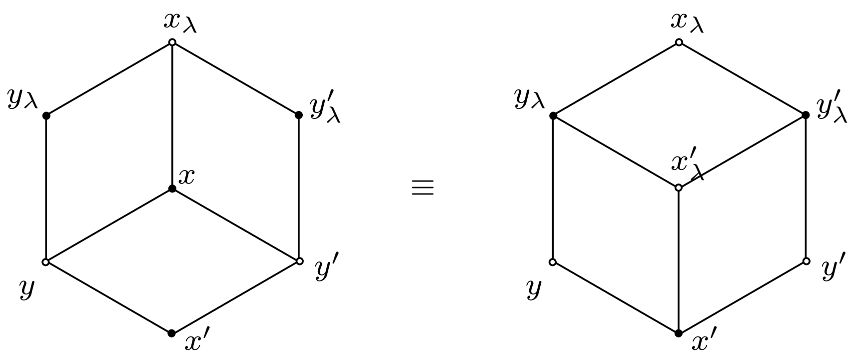

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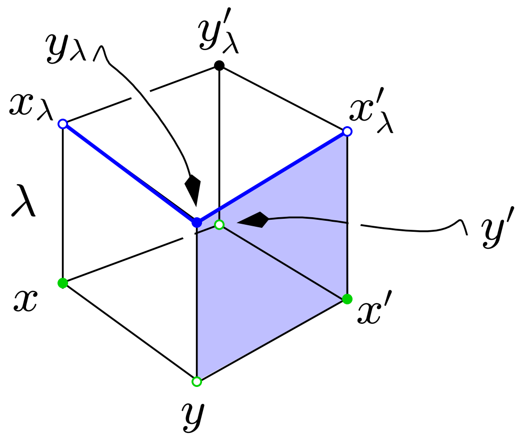


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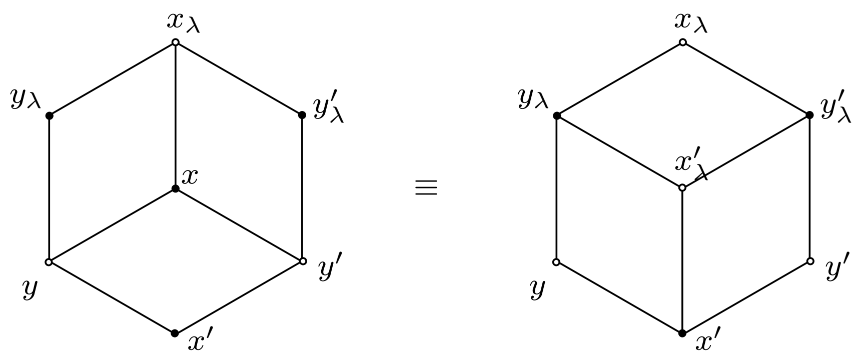

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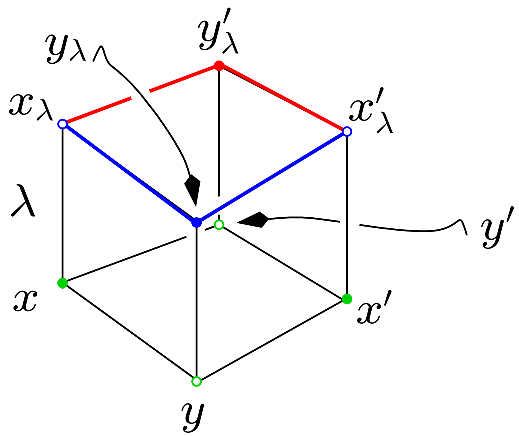


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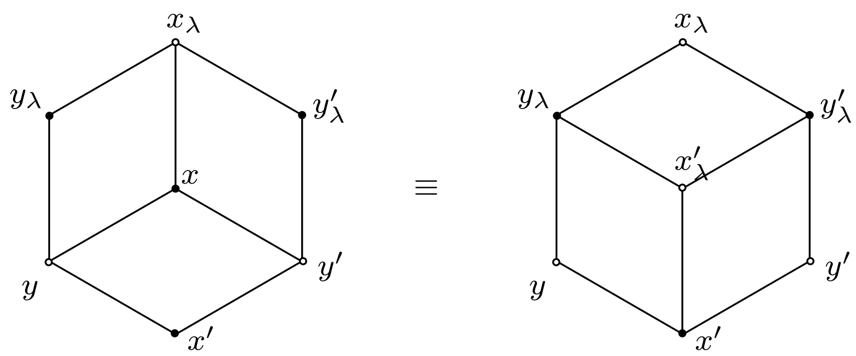

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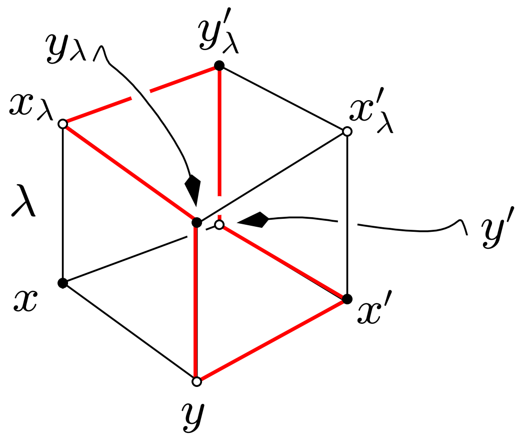


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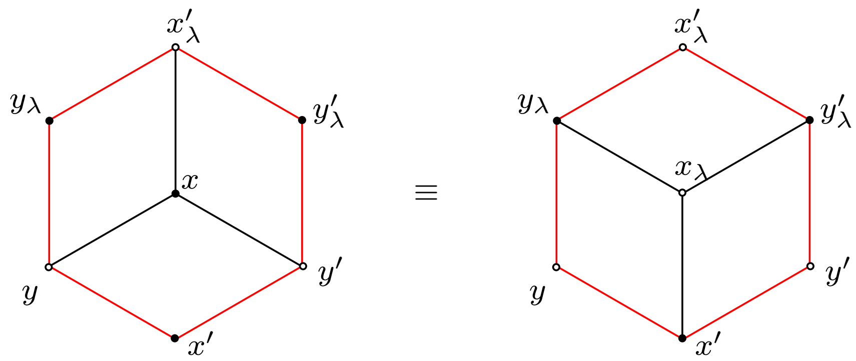
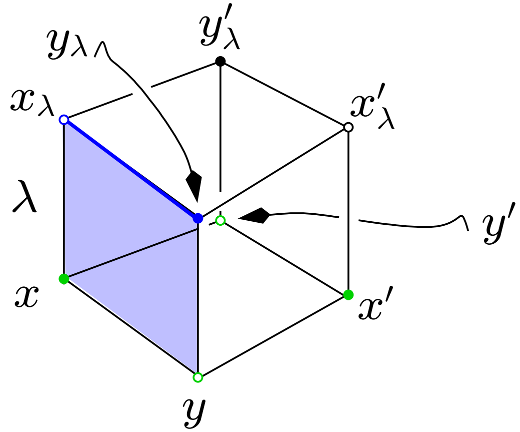


# Solitons

The vertical equation is

$$\frac{(f_\lambda(y) - f_\lambda(x))\,(f(y) - f(x))}{(f_\lambda(y) - f(y))\,(f_\lambda(x) - f(x))} = \quad q = \frac{(y-x)^2}{\lambda^2}$$

$$i\,\rho = \frac{y-x+\lambda}{x-y+\lambda}$$

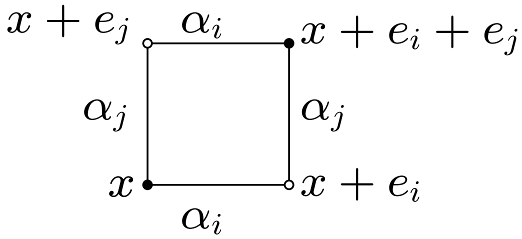



With the change of variables $g = \ln f$, the continuous limit ($y = x + \epsilon$) is :

$$\sqrt{g'_\lambda \times g'} = \frac{2}{\lambda}\,\sinh\frac{g_\lambda - g}{2}.$$

# Isomonodromic solutions, moving frame

The Bäcklund transformation allows to define a notion of discrete holomorphy in $\mathbb{Z}^d$, for $d > 1$ finite, equipped with rapidities $(\alpha_i)_{1 \le i \le d}$.




$$\rho = \frac{\alpha_i + \alpha_j}{\alpha_i - \alpha_j} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

$$q = \frac{\alpha_i^2}{\alpha_j^2}$$

$$f(x+e_i+e_j) = L_i(x+e_j) f(x+e_j), \qquad L_i(x+e_j) \in \mathbb{P}GL_2(\mathbb{C})$$

$$L_i(x;\lambda) = \begin{pmatrix} \lambda + \alpha_i & -2\alpha_i (f(x+e_i) + f(x)) \\ 0 & \lambda - \alpha_i \end{pmatrix}$$

The *moving frame* $\Psi(\cdot, \lambda) : \mathbb{Z}^d \to GL_2(\mathbb{C})[\lambda]$ for a prescribed $\Psi(0;\lambda)$ :

$$\Psi(x+e_i;\lambda) = L_i(x;\lambda)\Psi(x;\lambda).$$

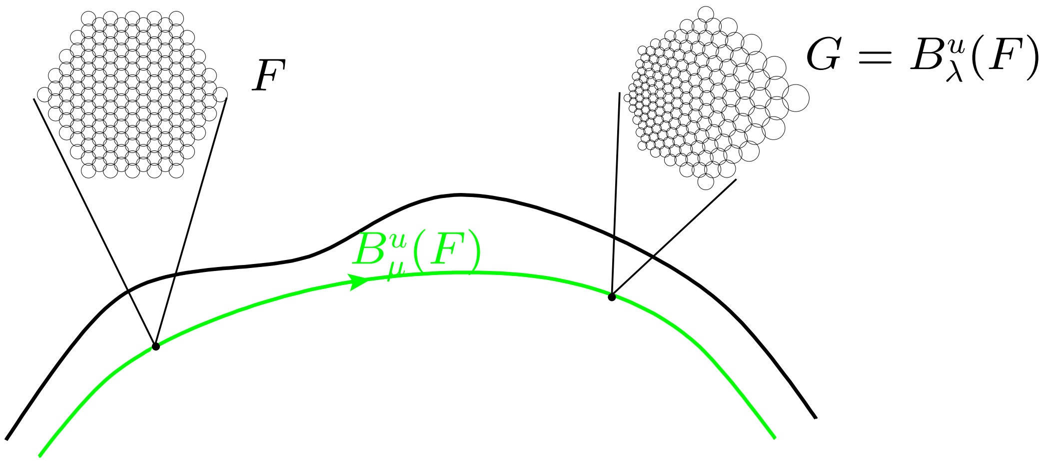
Define $A(\cdot;\lambda) : \mathbb{Z}^d \to GL_2(\mathbb{C})[\lambda]$ by $A(x;\lambda) = \dfrac{d\Psi(x;\lambda)}{d\lambda}\Psi^{-1}(x;\lambda)$.

They satisfy the recurrent relation

$$A(x+e_k;\lambda) = \frac{dL_k(x;\lambda)}{d\lambda} L_k^{-1}(x;\lambda) + L_k(x;\lambda) A(x;\lambda) L_k^{-1}(x;\lambda).$$

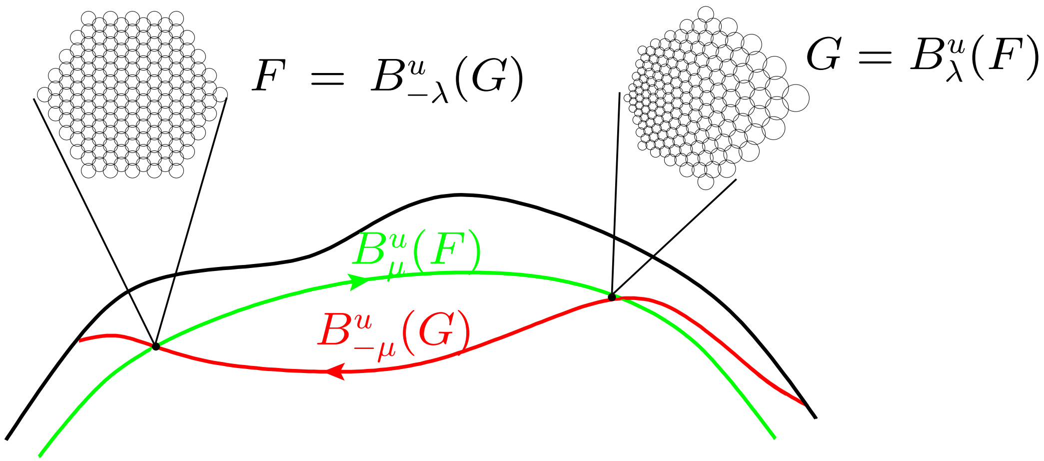
A discrete holomorphic function $f : \mathbb{Z}^d \to \mathbb{C}$ is called **isomonodromic**, if, for some choice of $A(0;\lambda)$, the matrices $A(x;\lambda)$ are meromorphic in $\lambda$, with poles whose positions and orders do not depend on $x \in \mathbb{Z}^d$. Isomonodromic solutions can be constructed with prescribed boundary conditions. Example : the **Green function**.

# The exponential




Two-parameter analytic family of CR-preserving maps, the initial condition $u$ at a given origin, and the parameter $\lambda$ of the Bäcklund transformation. It induces a linear map between the tangent spaces. It is not injective, its kernel is named the **discrete exponential** $\exp(:\lambda\colon F)$. Varying the parameter, they form a **basis** of the tangent space, the space of linear discrete holomorphic functions.

## The exponential

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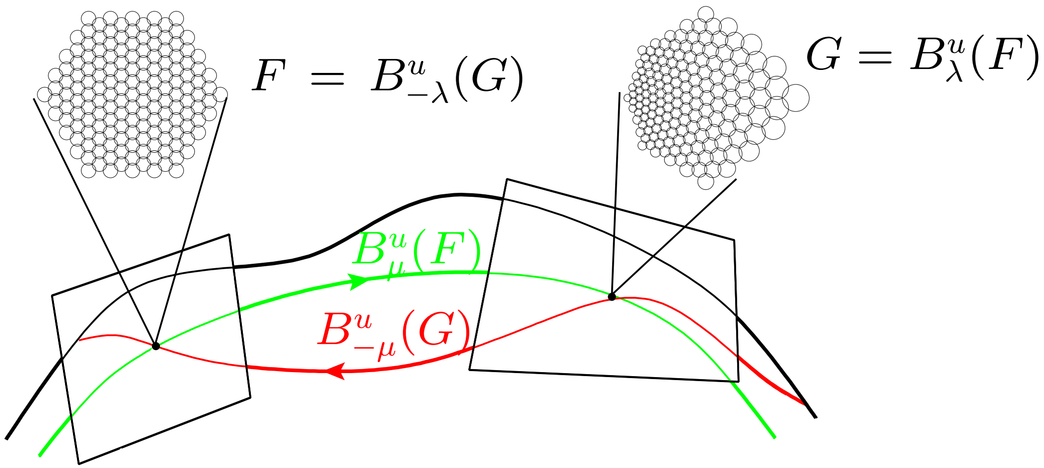


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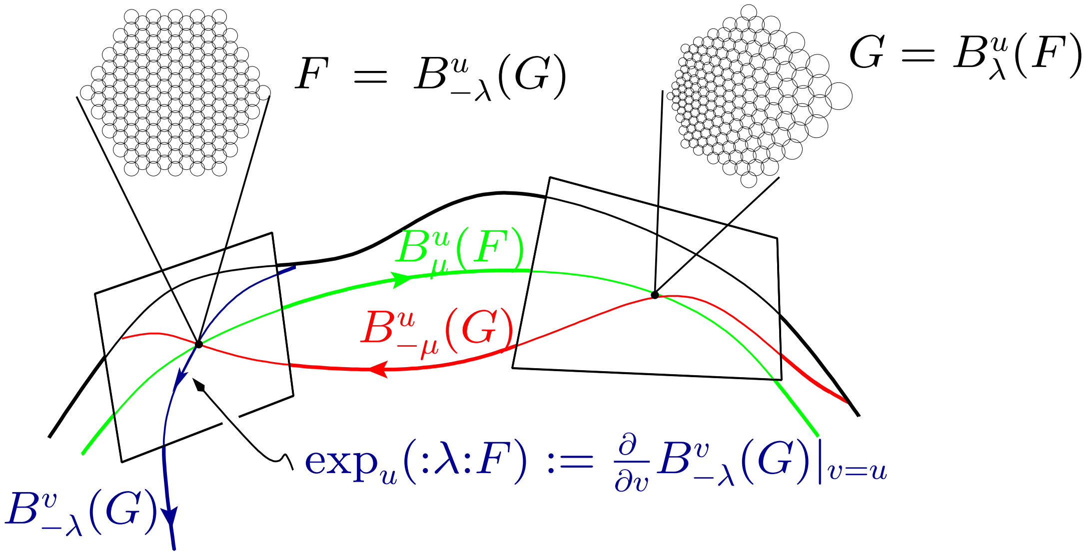


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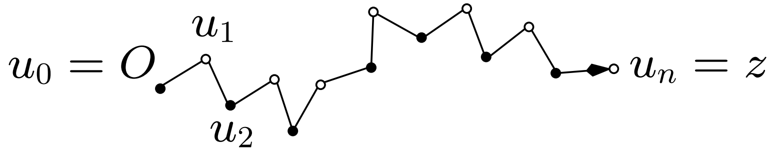
Example : The lozenges.




$$\exp(:\!\lambda\!: z) = \prod_k \frac{1 + \frac{\lambda}{2}(u_k - u_{k-1})}{1 - \frac{\lambda}{2}(u_k - u_{k-1})}.$$

It is a generalization of the formula

$$\exp(\lambda\, z) = \left(1 + \frac{\lambda\, z}{n}\right)^n + O(\frac{z^2}{n}) = \left(\frac{1 + \frac{\lambda\, z}{2\, n}}{1 - \frac{\lambda\, z}{2\, n}}\right)^n + O(\frac{z^3}{n^2}).$$

The Green function on any rhombic map is

**Theorem 1 (Richard Kenyon)**

$$G(O, x) = -\frac{1}{8\, \pi^2\, i} \oint_C \exp(:\!\lambda\!: x)\, \frac{\log \frac{\delta}{2} \lambda}{\lambda}\, d\lambda$$

$$\Delta G(O, x) = \delta_{O,x}, \quad G(O, x) \sim_{x \to \infty} \begin{cases} \log |x| & \textit{for black vertices,} \\ i\, \arg(x) & \textit{for white vertices.} \end{cases}$$

The derivative of the exponential with respect to the parameter $\lambda$ yield

$$\frac{\partial^k}{\partial \lambda^k} \exp(:\lambda\colon z) =: Z^{:k:} \exp(:\lambda\colon z)$$

and $\lambda = 0$ defines the discrete polynomials $Z^{:k:}$ which in turn give $\exp(:\lambda\colon z) = \sum \frac{Z^{:k:}}{k!}$, absolutely convergent for $\lambda \cdot \delta < 1$.

In this case, the primitive of a holomorphic function is itself holomorphic :

$$\int_{(x,y)} f\, dZ := \frac{f(x) + f(y)}{2}(y - x).$$

We can solve some "differential equations", for example the polynomials fulfill

$$Z^{:k+1:} = (k+1) \int Z^{:k:}\, dZ,$$

and the exponential

$$\exp(:\lambda\colon z) = \frac{1}{\lambda} \int \exp(:\lambda\colon z)\, dZ.$$

There is a duality : with $\varepsilon = \pm 1$ on black and whites, $f^\dagger = \varepsilon\, \bar{f}$ is again holomorphic.

We define the derivative $f'$ of a holomorphic function $f$ as

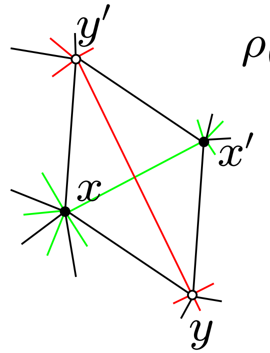
$$f'(z) := \frac{4}{\delta^2} \left( \int_O^z f^\dagger dZ \right)^\dagger + \lambda\, \varepsilon,$$

it verifies $f = \int f' dZ$.

Most of the tools of the continuous theory are recovered in the discrete linear case, harmonicity, Hodge decomposition, poles, Riemann-Roch theorem, period matrix, Cauchy integral formulae, Abel's map in the Jacobian, Riemann bilinear relations and a continuous limit theorem.

## Discrete hamonicity

- Hodge star : $\quad * : C^k \to C^{2-k}$, $\qquad \int_{(y,y')} * \alpha := \rho_{(x,x')} \int_{(x,x')} \alpha.$
- Discrete Laplacian :
  $d^* := - * d *, \quad \Delta := d\, d^* + d^* d,$ $\qquad \Delta f(x) = \sum \rho_{(x,x_k)} (f(x) - f(x_k)).$
- Hodge decomposition :
  $C^k = \mathrm{Im} d \oplus^\perp \mathrm{Im} d^* \oplus^\perp \mathrm{Ker}\ \Delta,$ $\qquad \mathrm{Ker}\ \Delta_1 = C^{(1,0)} \oplus^\perp C^{(0,1)}.$
- Weyl's lemma :
  $\Delta f = 0 \iff \iint f * \Delta g = 0, \quad \forall g$ compact.
- Green's identity :
  $\iint_D (f * \Delta g - g * \Delta f) = \oint_{\partial D} (f * dg - g * df).$




$\rho_{(x,x')} = -i \frac{y'-y}{x'-x}$

*cf* Wardetzky, Polthier, Glickenstein, Novikov, Wilson

## Meromorphic forms

$$\alpha \in C^1 \text{ is holomorphic } \iff \begin{cases} *\alpha = -i\alpha & \text{of type (1,0)} \\ d\alpha = 0 & \text{closed.} \end{cases} \qquad \textit{i.e.} \int_{(y,y')} \alpha = i\rho_{(x,x')} \int_{(x,x')} \alpha$$

- $\alpha$ of type (1,0) has a pole of order $1$ at $v$ if its residue $\quad \mathrm{Res}_v\, \alpha := \frac{1}{2i\pi} \oint_{\partial v^*} \alpha \neq 0$.
- If $\alpha$ is not of type (1,0) on $(x, y, x', y')$ it has there a pole of order $> 1$.
- Discrete Riemann-Roch theorem : Existence of forms with prescribed poles and holonomies.
- Green function and potential, Cauchy integral formula : $\quad \oint_{\partial D} f \cdot \nu_{x,y} = 2i\pi \frac{f(x)+f(y)}{2}$.
- Period matrix, Jacobian, Abel's map, Riemann's bilinear relations.
- In the rhombic case : integration of functions, derivation, explicit basis (exponentials and polynomials), continuous limit theorem.

The $L^2$ norm of the 1-form $df$ is called the **Dirichlet** energy

$$E_D(f) := \|df\|^2 = (df,\, df) = \frac{1}{2} \sum_{(x,x') \in \Lambda_1} \rho(x,x') \left| f(x') - f(x) \right|^2$$
$$= \frac{E_D(f|_\Gamma) + E_D(f|_{\Gamma^*})}{2}.$$

The **conformal energy** of a map measures its conformality defect

$$E_C(f) := \tfrac{1}{2} \|df - i * df\|^2.$$

They are related through

$$\begin{aligned} E_C(f) &= \tfrac{1}{2} \left( df - i * df,\, df - i * df \right) \\ &= \tfrac{1}{2} \|df\|^2 + \tfrac{1}{2} \|-i * df\|^2 + \operatorname{Re}(df,\, -i * df) \\ &= \|df\|^2 + \operatorname{Im} \iint_{\diamondsuit_2} df \wedge \overline{df} \\ &= E_D(f) - 2\mathcal{A}(f) \end{aligned}$$

where the algebraic area covered by the function is

$$\mathcal{A}(f) := \frac{i}{2} \iint_{\diamondsuit_2} df \wedge \overline{df}$$

For a face $(x,y,x',y')\in\diamondsuit_2$, the algebraic area of the oriented quadrilateral $\Big(f(x),f(x'),f(y),f(y')\Big)$ is

$$\begin{aligned}\iint\limits_{(x,y,x',y')} df\wedge\overline{df} &= i\,\mathrm{Im}\Big((f(x')-f(x))\overline{(f(y')-f(y))}\Big)\\ &= -2i\mathcal{A}\Big(f(x),f(x'),f(y),f(y')\Big)\end{aligned}$$

In the continuous,

$$\text{for } f(z+z_0)=f(z_0)+z\times(\partial f)(z_0)+\bar z\times(\bar\partial f)(z_0)+o(|z|),$$

$$(\partial f)(z_0)=\lim_{\gamma\to z_0}\frac{i}{2\mathcal{A}(\gamma)}\oint_\gamma f d\bar z,\qquad\qquad (\bar\partial f)(z_0)=-\lim_{\gamma\to z_0}\frac{i}{2\mathcal{A}(\gamma)}\oint_\gamma f dZ,$$

over a small loop $\gamma$ around $z_0$.

It leads to the following definitions in the discrete setup :

$$\begin{aligned}\partial: C^0(\diamondsuit) &\to C^2(\diamondsuit)\\ f &\mapsto \partial f = \Big[(x,y,x',y') \mapsto -\tfrac{i}{2\mathcal{A}(x,y,x',y')}\oint\limits_{(x,y,x',y')} f d\bar Z\Big]\\ &\qquad = \tfrac{(f(x')-f(x))(\bar y'-\bar y)-(\bar x'-\bar x)(f(y')-f(y))}{(x'-x)(\bar y'-\bar y)-(\bar x'-\bar x)(y'-y)},\end{aligned}$$

$$\begin{aligned}\bar\partial: C^0(\diamondsuit) &\to C^2(\diamondsuit)\\ f &\mapsto \bar\partial f = \Big[(x,y,x',y') \mapsto -\tfrac{i}{2\mathcal{A}(x,y,x',y')}\oint\limits_{(x,y,x',y')} f dZ\Big]\\ &\qquad = \tfrac{(f(x')-f(x))(y'-y)-(x'-x)(f(y')-f(y))}{(x'-x)(\bar y'-\bar y)-(\bar x'-\bar x)(y'-y)}.\end{aligned}$$

A holomorphic function $f$ verifies $\bar{\partial} f \equiv 0$ and (with $Z(u)$ noted simply $u$)

$$\partial f(x, y, x', y') = \frac{f(y') - f(y)}{y' - y} = \frac{f(x') - f(x)}{x' - x}.$$

The Jacobian $J = |\partial f|^2 - |\bar{\partial} f|^2$ relates the area :

$$\iint_{(x,y,x',y')} df \wedge \overline{df} = J \iint_{(x,y,x',y')} dZ \wedge \overline{dZ}.$$

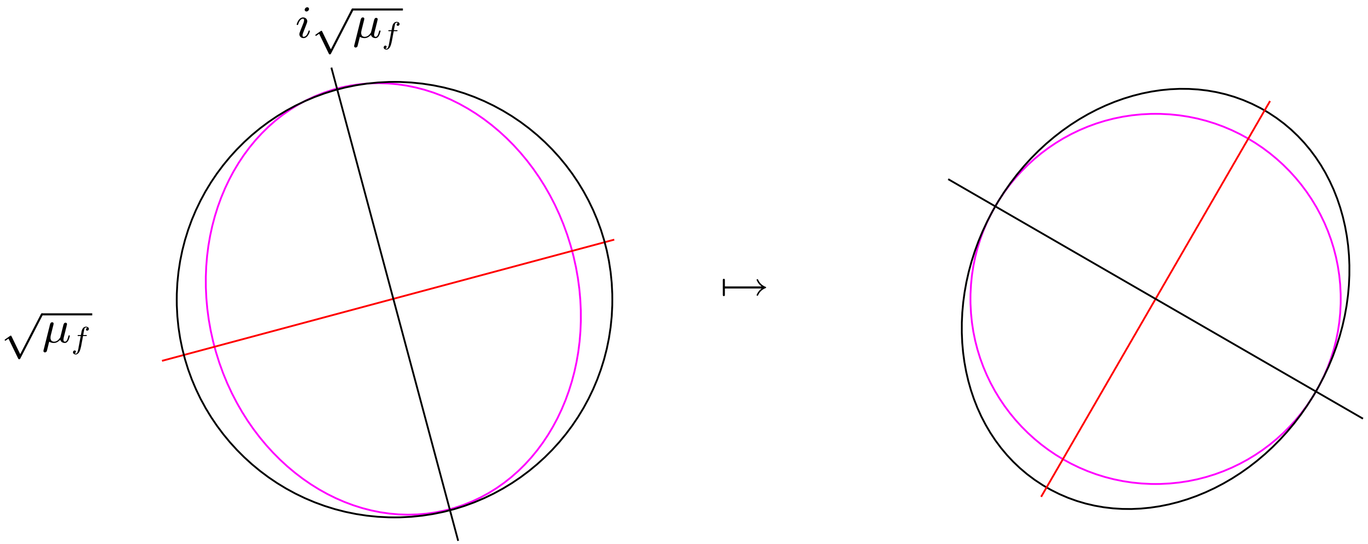
# Quasi-conformal maps

For a general function, define the *dilatation* coefficient

$$D_f := \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}$$

$D_f \geq 1$ for $|f_{\bar{z}}| \leq |f_z|$ (quasi-conformal). Can be written in term of the *complex dilatation* :

$$\mu_f = \frac{f_{\bar{z}}}{f_z}$$



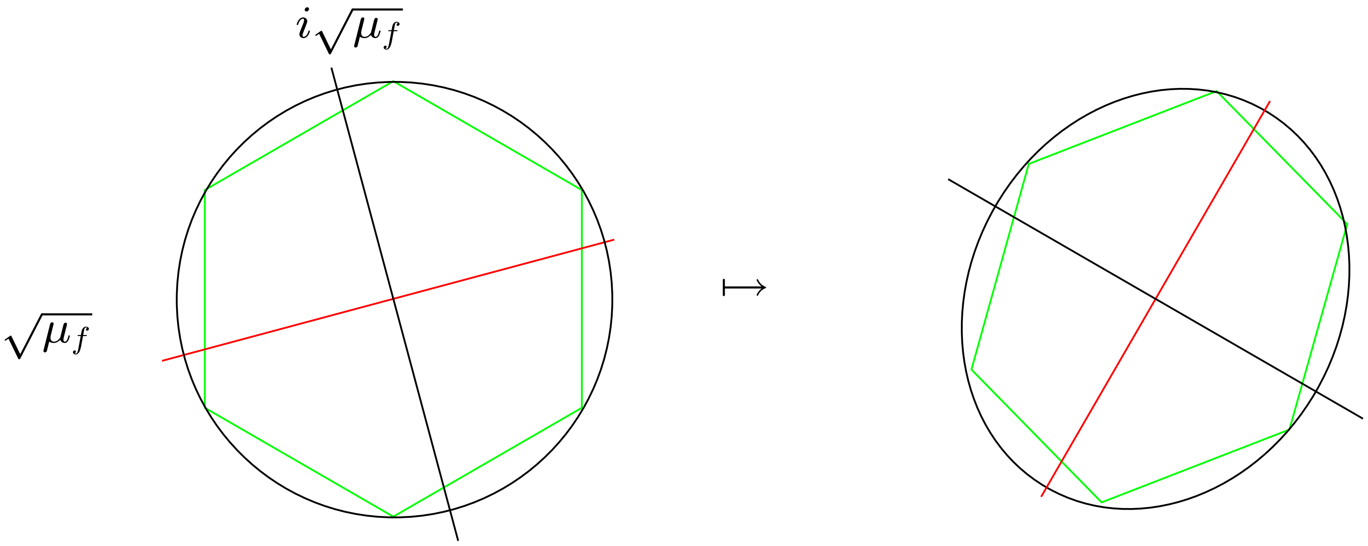
# Quasi-conformal maps

For a general discrete function, define the *dilatation* coefficient

$$D_f := \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}$$

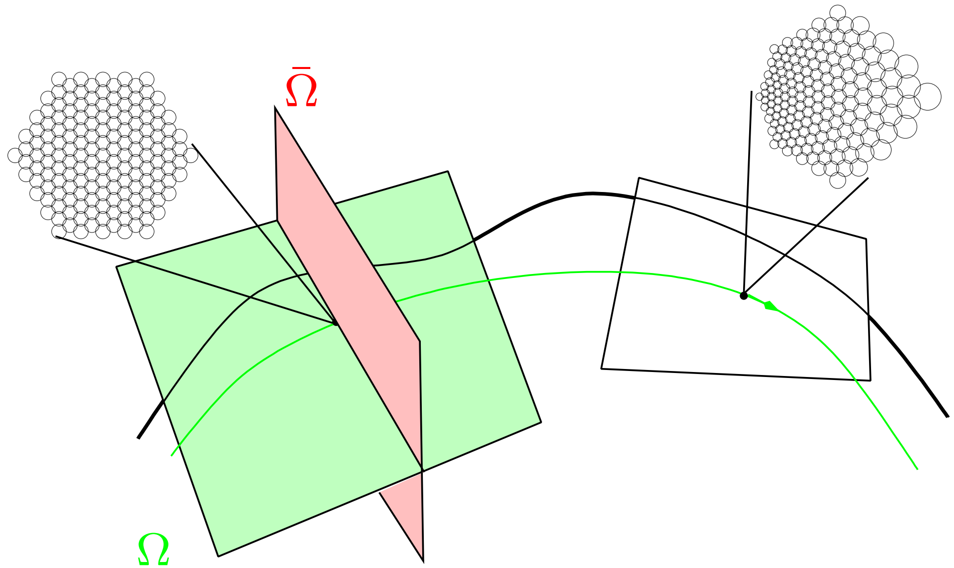
$D_f \geq 1$ for $|f_{\bar{z}}| \leq |f_z|$ (quasi-conformal). Can be written in term of the *complex dilatation* :

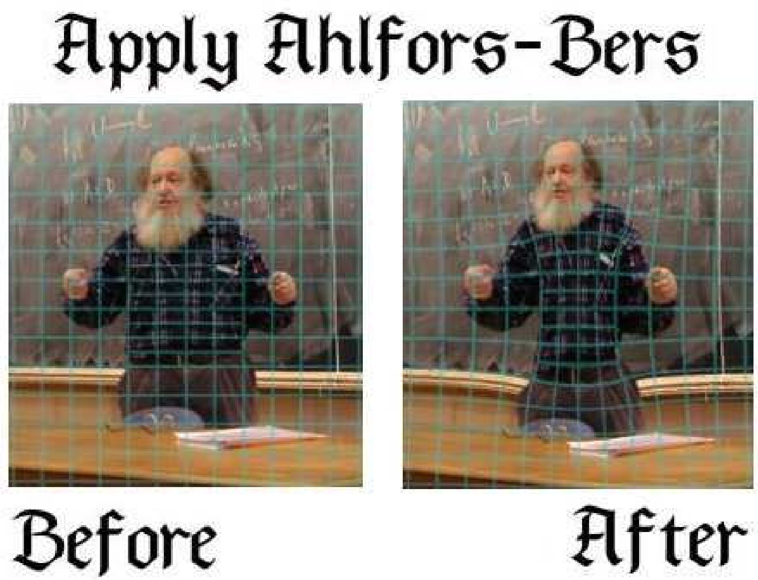
$$\mu_f = \frac{f_{\bar{z}}}{f_z} = \frac{(f(x') - f(x))(y' - y) - (x' - x)(f(y') - f(y))}{(f(x') - f(x))(\bar{y}' - \bar{y}) - (\bar{x}' - \bar{x})(f(y') - f(y))}$$



## Teichmüller space?

For a given combinatorics the fixed cross-ratio subspaces foliate the space of maps. $\Omega$ stays on the same discrete Riemann surface, $\bar{\Omega}$ changes the discrete conformal structure.



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# Thank you.