A Universality Theorem for Realization Spaces of Maps

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A universality theorem for maps in $\mathbb{R}^3$ is shown, stating essentially that every semialgebraic set can occur as a realization space of some map (with a distinguished set of vertices); more precisely:

Theorem. (Universality Theorem for Maps)

Let $n \geq 0$ and G be a graph with vertex set $\{v_1,\ldots,v_5, w_1,\ldots,w_n\}$. Let $P \subseteq \mathbb{R}^{3n}$ be a semialgebraic set defined over $\mathbb{Q}$. Then there is a map $\mathfrak{M}$ (on some orientable manifold) containing G as an induced subgraph such that for each subfield $K \subseteq \mathbb{R}$ a straight standard[1] embedding f of G in K^3 can be extended to a polyhedral embedding (i.e. the facets are strictly convex) of $\mathfrak{M}$ in K^3 if and only if $(f(w_1),\ldots, f(w_n)) \in P$.

Corollaries.

(1) For each strict subfield L of the field of real algebraic numbers there is a map $\mathfrak{M}$ which can be polyhedrally embedded in $\mathbb{R}^3$ but not in L^3.

(2) The realizability problem for maps in $\mathbb{R}^3$ is polynomial time equivalent to the `Existential Theory of the Reals' and thus NP-hard.

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[1] 'standard' means that $v_1,\ldots,v_5$ are mapped onto some fixed given projective base