Yang-Baxter equation and discrete conformal symmetry

Vladimir Bazhanov

Department of Theoretical Physics
Research School of Physical Sciences and Engineering
The Australian National University

Discrete Differential Geometry,
Berlin, July, 15-19, 2007.

[work with V.Mangazeev and S.Sergeev]

Connection between statistical mechanics and discrete geometry

- Exactly solvable models:
  - Yang-Baxter equation, Yang (1967), Baxter (1972),
  - Faddeev-Volkov solution (1992-94)
- Discrete Riemann mapping theorem.
  - Thurston (1985), Rodin-Sullivan (1987), Stephenson (1987), He-Schramm (1998), ...
  - Discrete analytic functions: Bobenko-& Pinkall (1996), -& Suris (2002), -& Mercat and Suris.
  - Variational principle for circle patterns: Bobenko-Springborn (2002)
- Topological invariants, braid group, invariants of links, rhombic tilings . . .
  - "Z-invariant" lattices, Baxter (1989) and invariants of links, Jones (1987)
  - Planar embeddings of quad-graphs, Kenyon-Schlenker (2005)
- Hyperbolic geometry: volumes of polyhedra in the Lobachevskii 3-space
- Conformal Field Theory, Belavin-Polyakov-Zamolodchikov (1984), A discrete analog?

Yang-Baxter equation in statistical mechanics

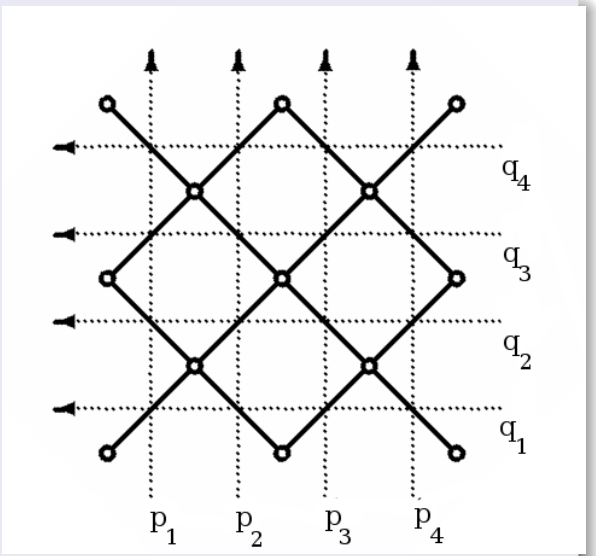
local "spins": $\sigma_i \in$ (set of values)

$$Z = \sum_{\{spins\}} e^{-E(\sigma)/T}, \qquad E(\{\sigma\}) = \sum_{(ij)\in edges} \epsilon(\sigma_i, \sigma_j),$$

$$W(\sigma_i, \sigma_j) = e^{-\epsilon(\sigma_i,\sigma_j)/T}$$

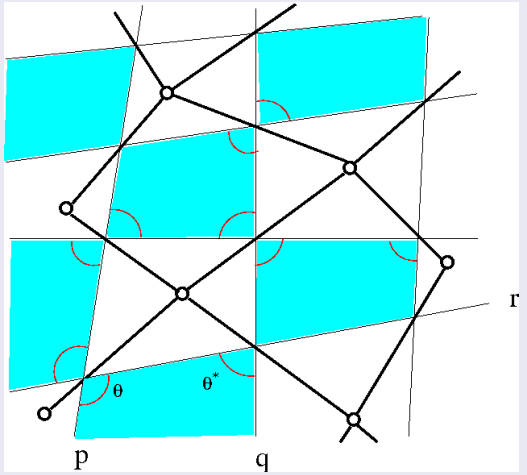
$$Z = \sum_{\{spins\}} \prod_{(ij)\in edges} W(\sigma_i, \sigma_j) = \mathrm{Trace}\,(\mathbf{T})^m,$$

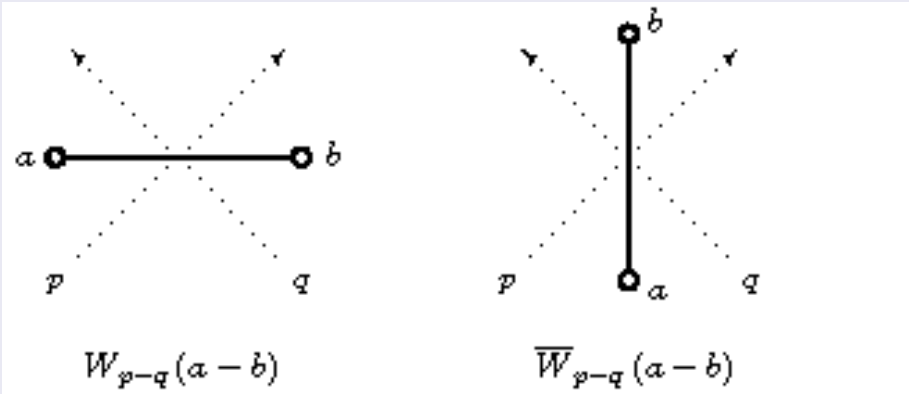
Hard to calculate if number of edges, $N \to \infty$



- Ising model, dimers, . . . , $\Rightarrow$ free fermions. "Gaussian models", reduce to diagonalization of a quadratic form, Pfaffians and determinants.
- The Boltzmann weights satisfy the Yang-Baxter equation. Commuting transfer-matrices

$$Z = \mathrm{Trace}\,(\mathbf{T_{q_1}}\,\mathbf{T_{q_2}} \cdots \mathbf{T_{q_m}}), \quad [\mathbf{T_q}, \mathbf{T_{q'}}] = 0.$$

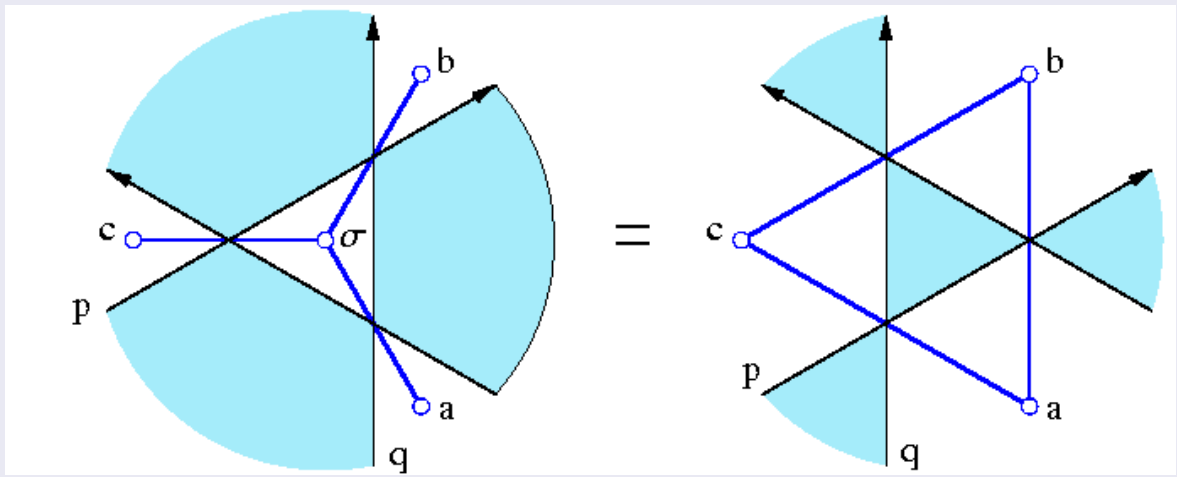




$$\theta = p - q, \quad \sigma, a, b \in \mathbb{R}$$

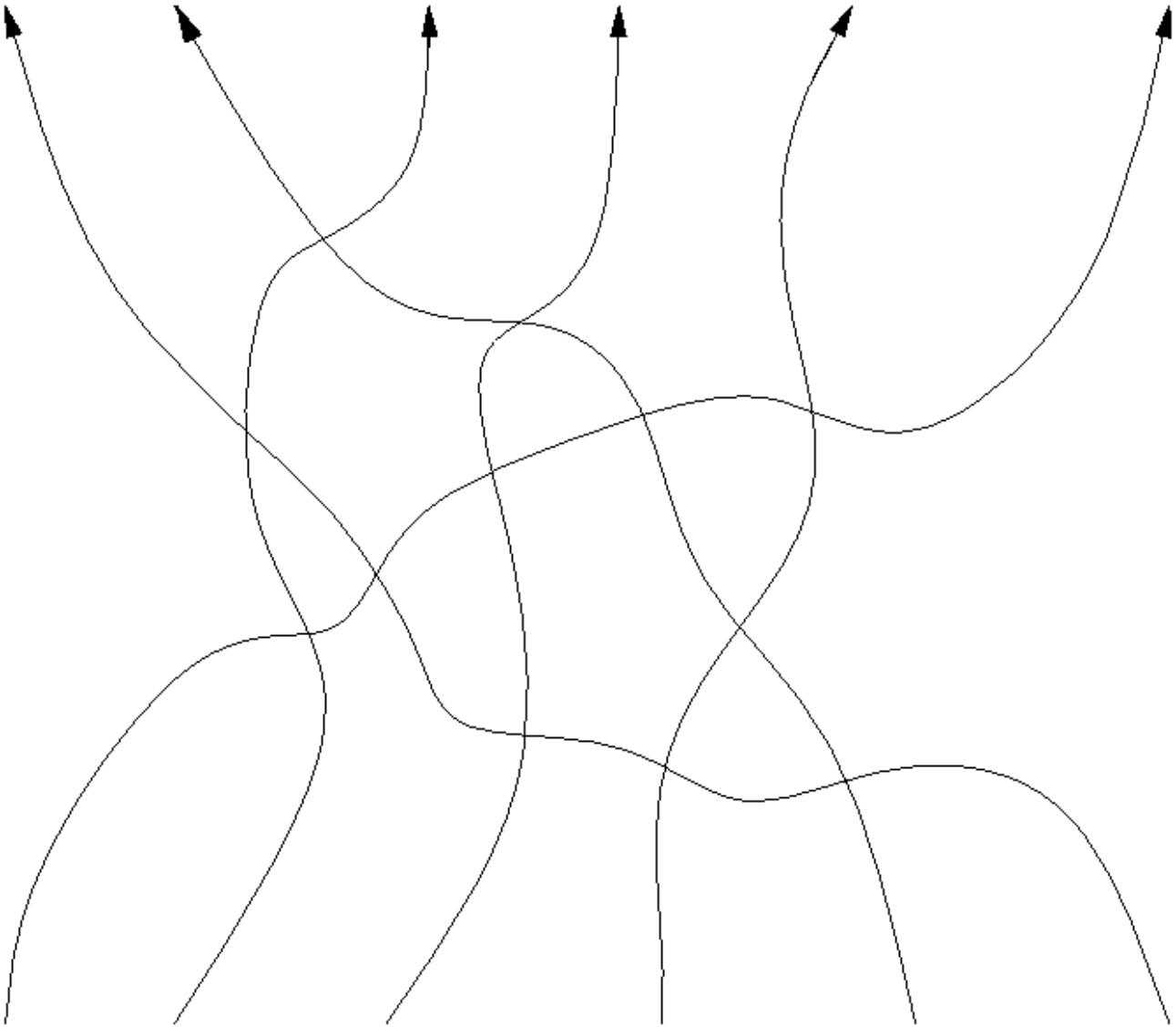
$$\overline{W}_\theta(a-b) = W_{\pi-\theta}(a-b)$$

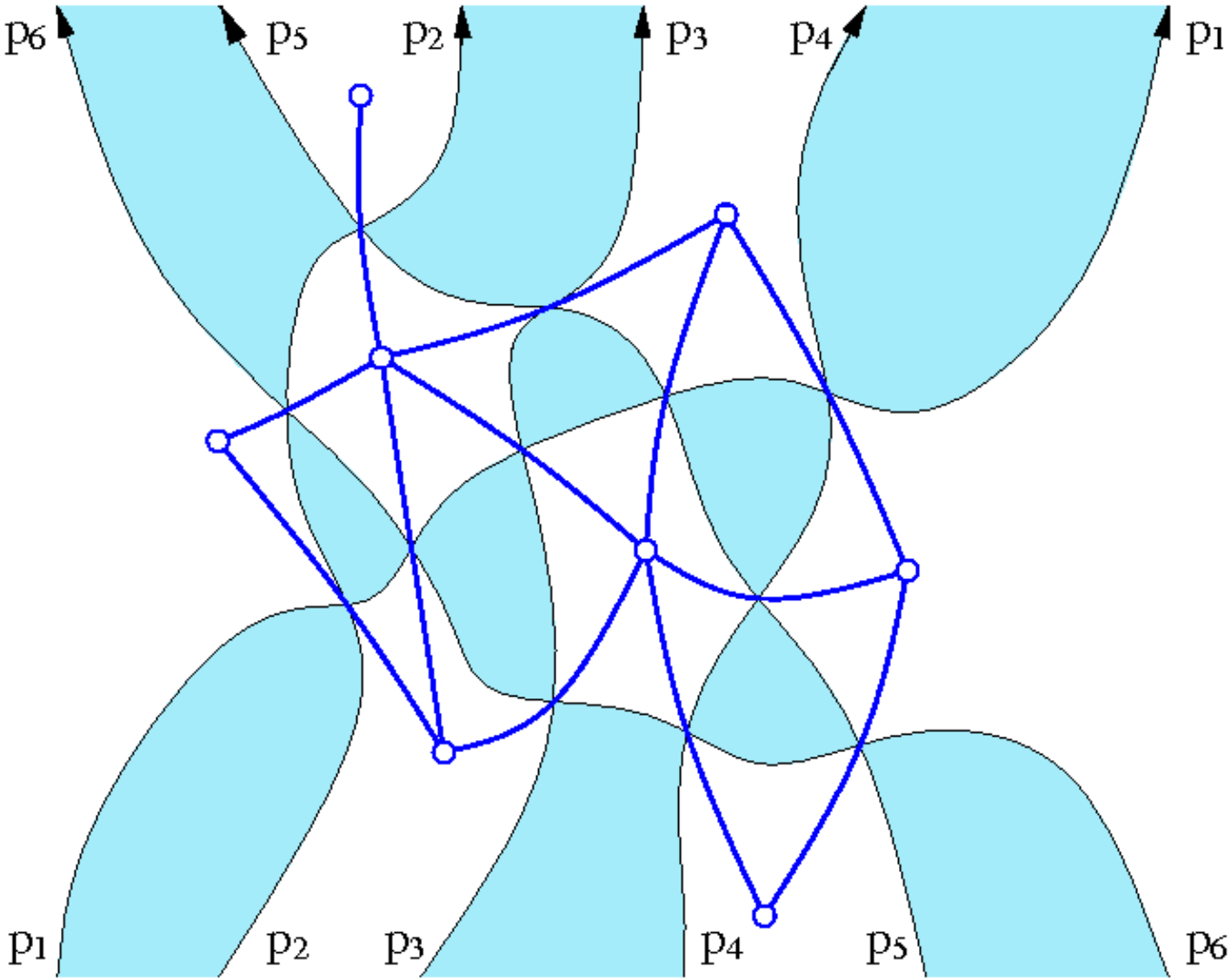
Yang-Baxter equation



Star-triangle relation:

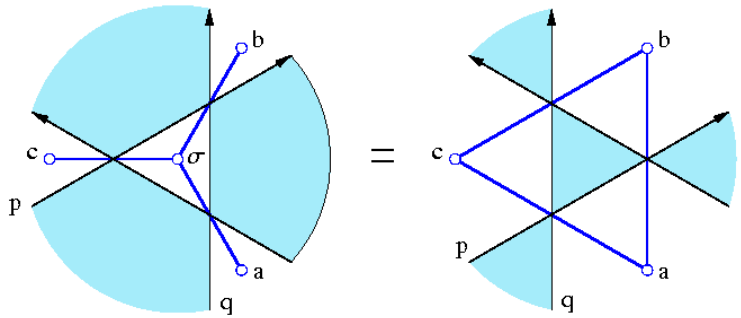
$$\int_{\mathbb{R}} d\sigma\ \overline{W}_{q-r}(a-\sigma)\ W_{p-r}(c-\sigma)\ \overline{W}_{p-q}(\sigma-b) \ = \ W_{p-q}(c-a)\overline{W}_{p-r}(a-b)\ W_{q-r}(c-b)\ .$$

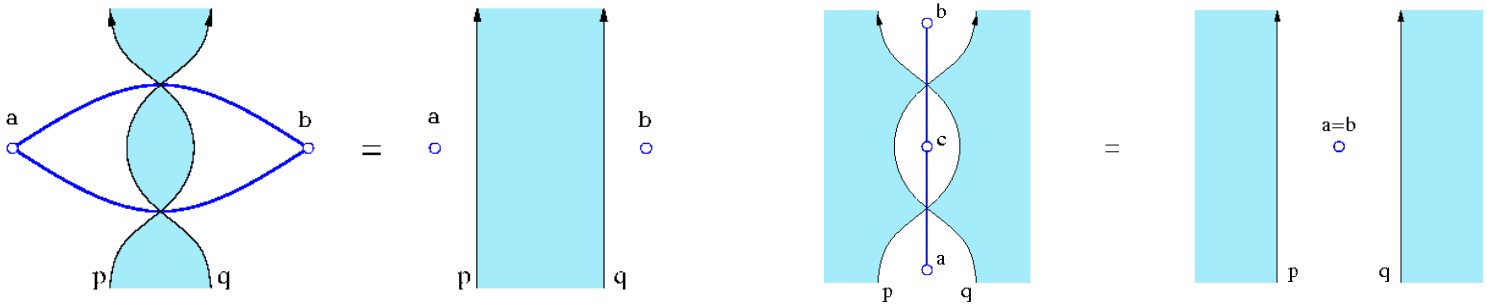



p6
p5
p2
p3
p4
p1
p1
p2
p3
p4
p5
p6

$$Z = \int \prod_{(ij)} W_{\theta_{(ij)}}(\sigma_i - \sigma_j) \prod_i d\sigma_i$$

Partition function Z possesses a remarkable invariance property: it remains unchanged by continuously deforming the rapidity lines (with their boundary position kept fixed)





Baxter (1979) factorization theorem. When $N \to \infty$

$$\log Z = \sum_{(ij)} f(\theta_{(ij)}) + O(\sqrt{N}).$$

Faddeev-Volkov model

$$W_\theta(s) = \frac{1}{F_\theta}\, \mathrm{e}^{2\eta\theta s}\, \frac{\varphi(s+\mathrm{i}\eta\theta/\pi)}{\varphi(s-\mathrm{i}\eta\theta/\pi)}\,, \qquad \overline{W}_\theta(s) = W_{\pi-\theta}(s)\,, \tag{1}$$

$$\varphi(z) \stackrel{\text{def}}{=} \exp\left(\frac{1}{4}\int_{\mathbb{R}+\mathrm{i}0} \frac{\mathrm{e}^{-2\mathrm{i}zw}}{\sinh(wb)\sinh(w/b)}\,\frac{dw}{w}\right)\,, \tag{2}$$

$$\Phi(z) \stackrel{\text{def}}{=} \exp\left(\frac{1}{8}\int_{\mathbb{R}+\mathrm{i}0} \frac{\mathrm{e}^{-2\mathrm{i}zw}}{\sinh(wb)\sinh(wb^{-1})\cosh(w(b+b^{-1}))}\,\frac{dw}{w}\right)\,, \tag{3}$$

$$\eta = (b+b^{-1})/2\,. \tag{4}$$

$$F_\theta \stackrel{\text{def}}{=} \mathrm{e}^{\mathrm{i}\eta^2\theta^2/\pi+\mathrm{i}\pi(1-8\eta^2)/24}\,\Phi(2\mathrm{i}\eta\theta/\pi)\,. \tag{5}$$

With this normalization the edge function $f(\theta) \equiv 0$, i.e.,

$$\log Z = O(\sqrt{N}), \qquad N \to \infty$$

The model related with quantum Liouville and sinh-Gordon equations and the modular double of quantum group

$$U_q(sl_2) \otimes U_{\tilde{q}}(sl_2), \qquad q = e^{i\pi b^2}, \quad \tilde{q} = e^{-i\pi/b^2},$$

Parameter $b^2 > 0$ plays the role of the Planck constant $\hbar$.

Quasi-classical expansion, $b \longrightarrow 0$

$$Z = \int e^{-\frac{1}{2\pi b^2}\mathcal{A}[\rho]} \prod_i \frac{d\rho_i}{2\pi b}$$

$$A(\theta|\rho) \;=\; A(\theta|-\rho) \;=\; \frac{1}{\mathrm{i}} \int_0^{\rho} \log\left(\frac{1+\mathrm{e}^{\xi+\mathrm{i}\theta}}{\mathrm{e}^{\xi}+\mathrm{e}^{\mathrm{i}\theta}}\right) d\xi \;.$$

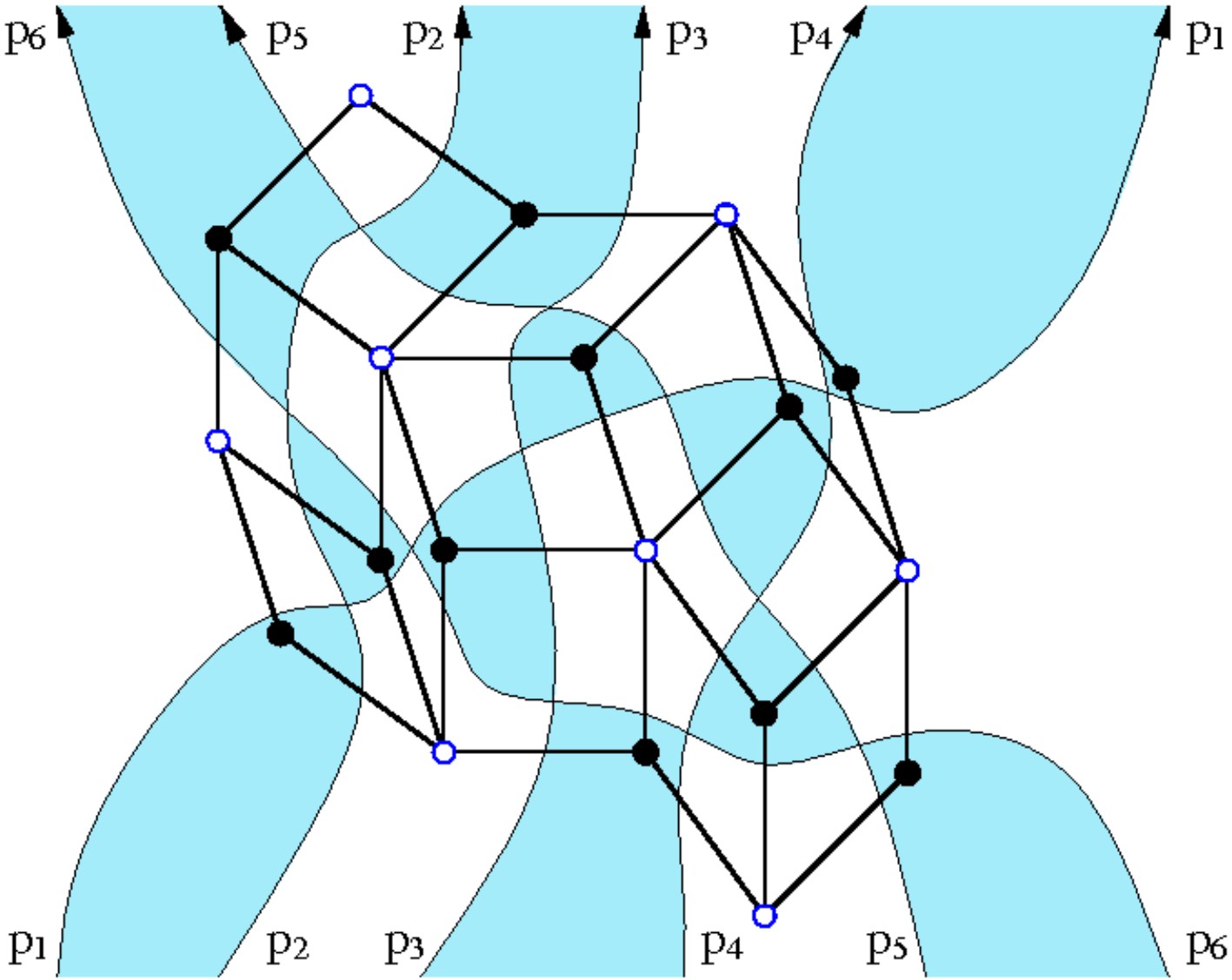
$$A(\theta|\rho) = i\, Li_2\left(\mathrm{e}^{\rho-\mathrm{i}\theta}\right) - i\, Li_2\left(\mathrm{e}^{\rho+\mathrm{i}\theta}\right) - \theta\rho, \qquad Li_2(x) = -\int_0^x \frac{\log(1-x)}{x}\, dx \;.$$

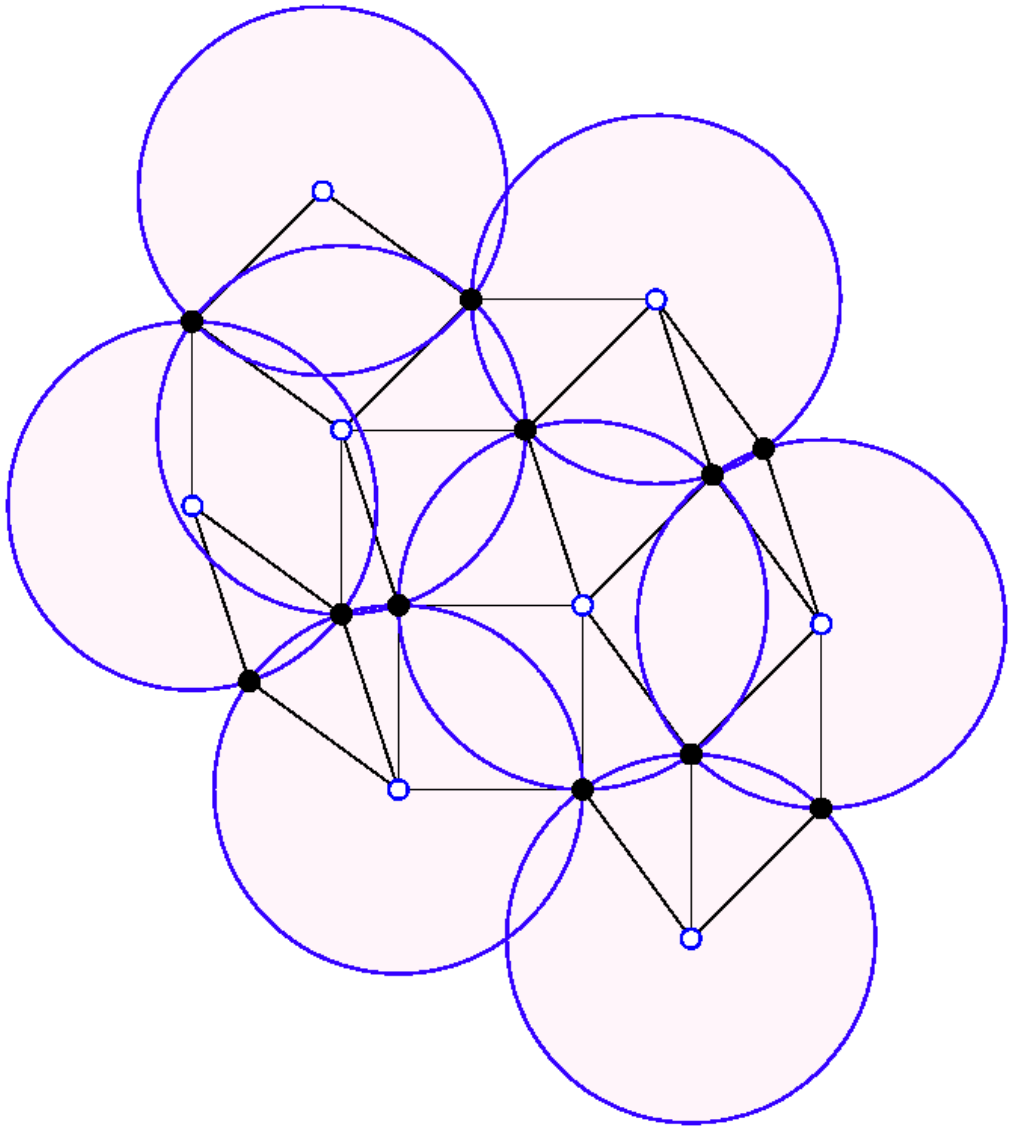
$$\mathcal{A}[\rho] = \sum_{(ij)\in E(\mathcal{G})} A\left(\theta_{(ij)} \,|\, \rho_i - \rho_j\right)$$

$$\frac{\partial \mathcal{A}[\rho]}{\partial \rho_i}\Big|_{\rho=\rho^{(cl)}} = 0, \Rightarrow \prod_{(ij)\in star(i)} \frac{\mathrm{e}^{\rho_j} + \mathrm{e}^{\rho_i+\mathrm{i}\theta_{(ij)}}}{\mathrm{e}^{\rho_i} + \mathrm{e}^{\rho_i+\mathrm{i}\theta_{(ij)}}} \;=\; 1 \;, \quad i =\in V_{int}(\mathcal{G}) \;.$$

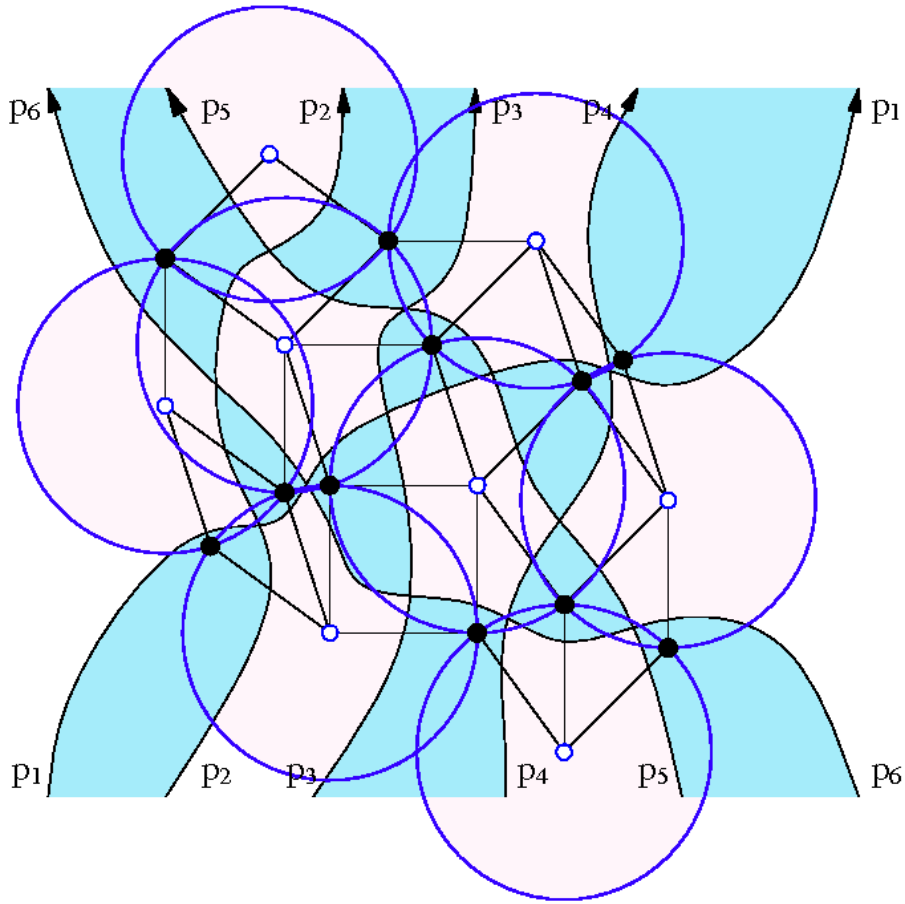
$\mathcal{A}[\rho]$ precisely coincide with Bobenko-Springborn circle packing action. Our results imply that this action possesses the "Z-invariance property" and that

$$\log Z = -\frac{1}{2\pi b^2}\mathcal{A}[\rho^{(cl)}] - \frac{1}{2}\log\det\left\|\frac{\partial^2 \mathcal{A}[\rho]}{\partial\rho_i\partial\rho_j}\right\|_{\rho=\rho^{(cl)}} + \ldots = O(\sqrt{N}), \quad N \to \infty$$


p6
p5
p2
p3
p4
p1
p1
p2
p3
p4
p5
p6



Kenyon-Schlenker: rhombic tilings, Bobenko-Mercat-Suris: isoradial circle patterns, this talk: "rapidity graph" or a braid.

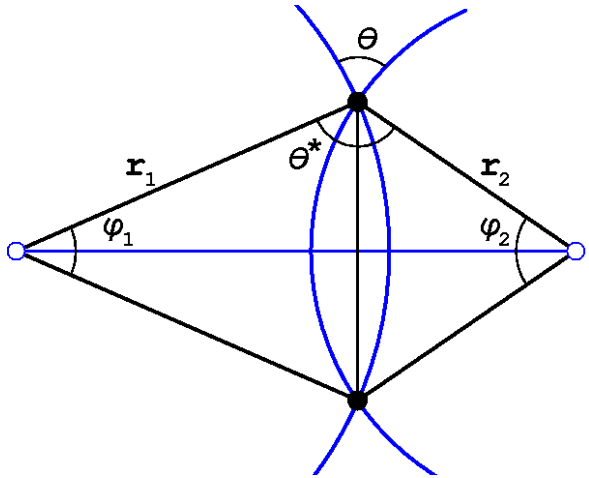


Sum rules

$$\sum_{(ij)\in star(i)} \theta_{(ij)} = 2\pi, \quad i \in V_{int}(\mathcal{G})$$

Discrete conformal transformations

- Continuous conformal transformations
  - (i) preserve angles
  - (ii) uniformly rescale all infinitesimal lengths (scale depends on a point)

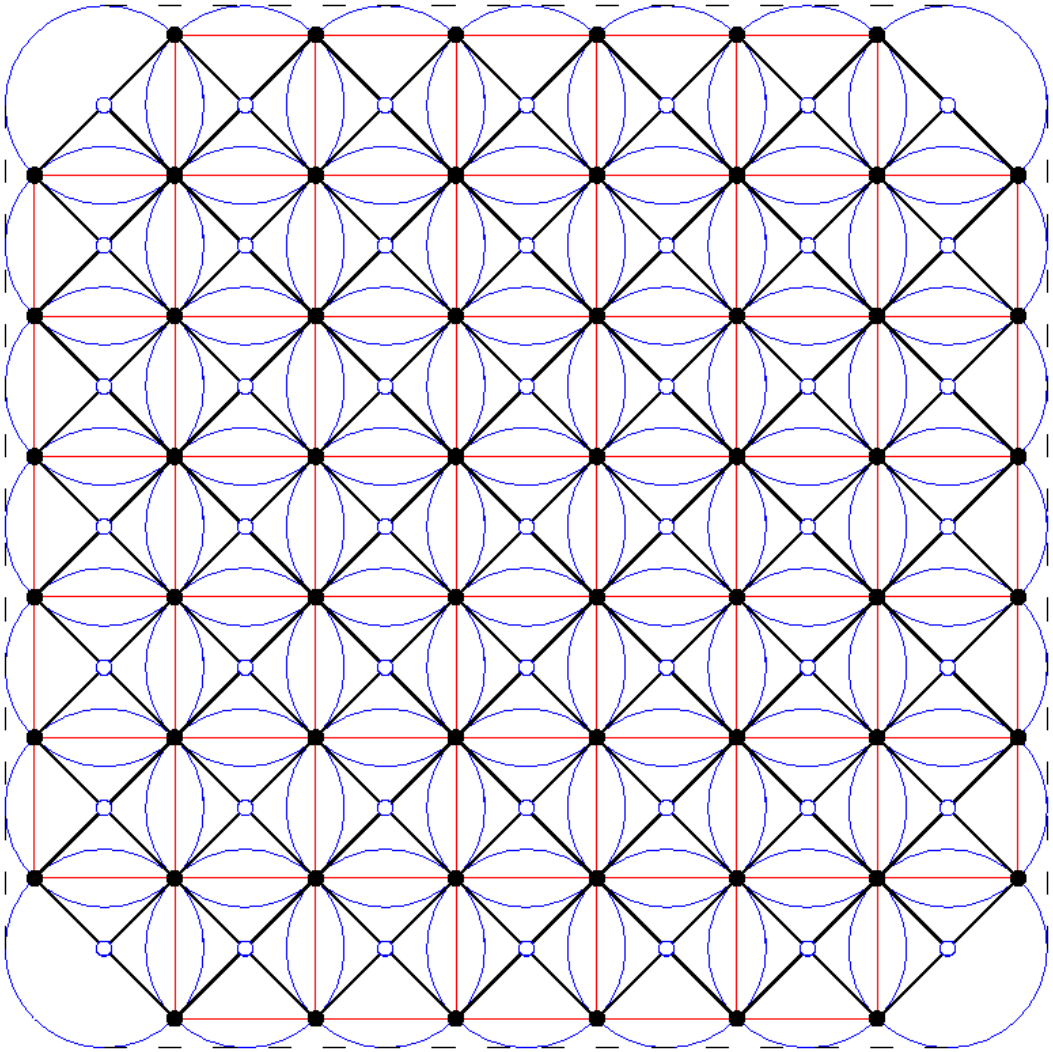


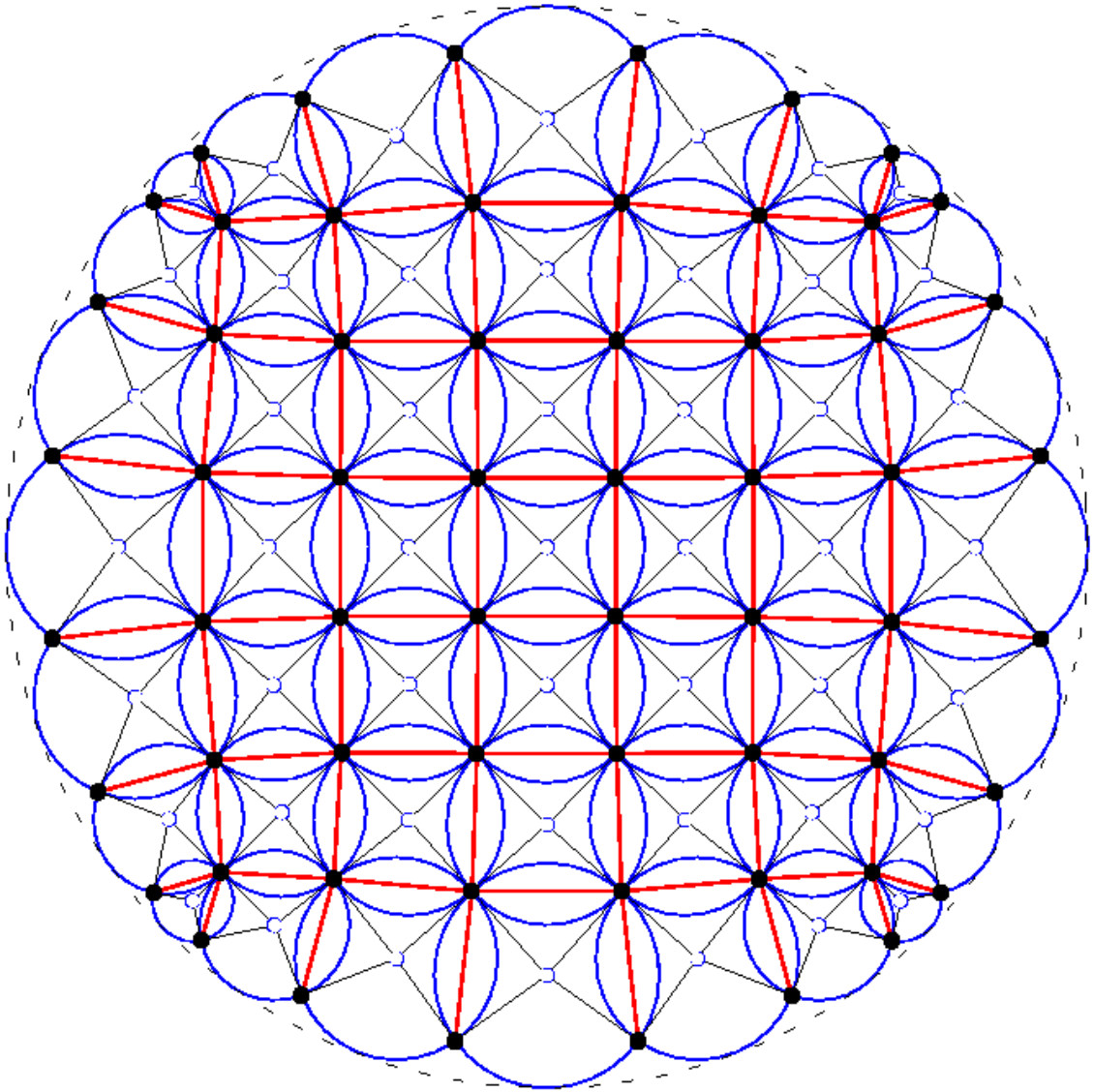
$$\varphi_1 = \frac{1}{i} \log \frac{r_1 + r_2 e^{i\theta}}{r_1 + r_2 e^{-i\theta}}, \tag{6}$$

Circle flower equations (cross ratio system)

$$\sum_{(ij)\in star(i)} \varphi_{(ij)} = 2\pi, \qquad i \in V_{int}(\mathcal{G}) \ . \tag{7}$$

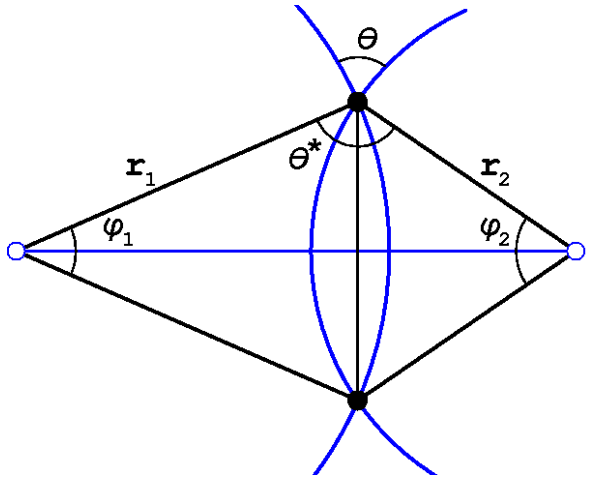
They are identical to the equation of motion in the Faddeev-Volkov model





Discrete conformal transformations

- Continuous conformal transformations
  - (i) preserve angles
  - (ii) uniformly rescale all infinitesimal lengths (scale depends on a point)

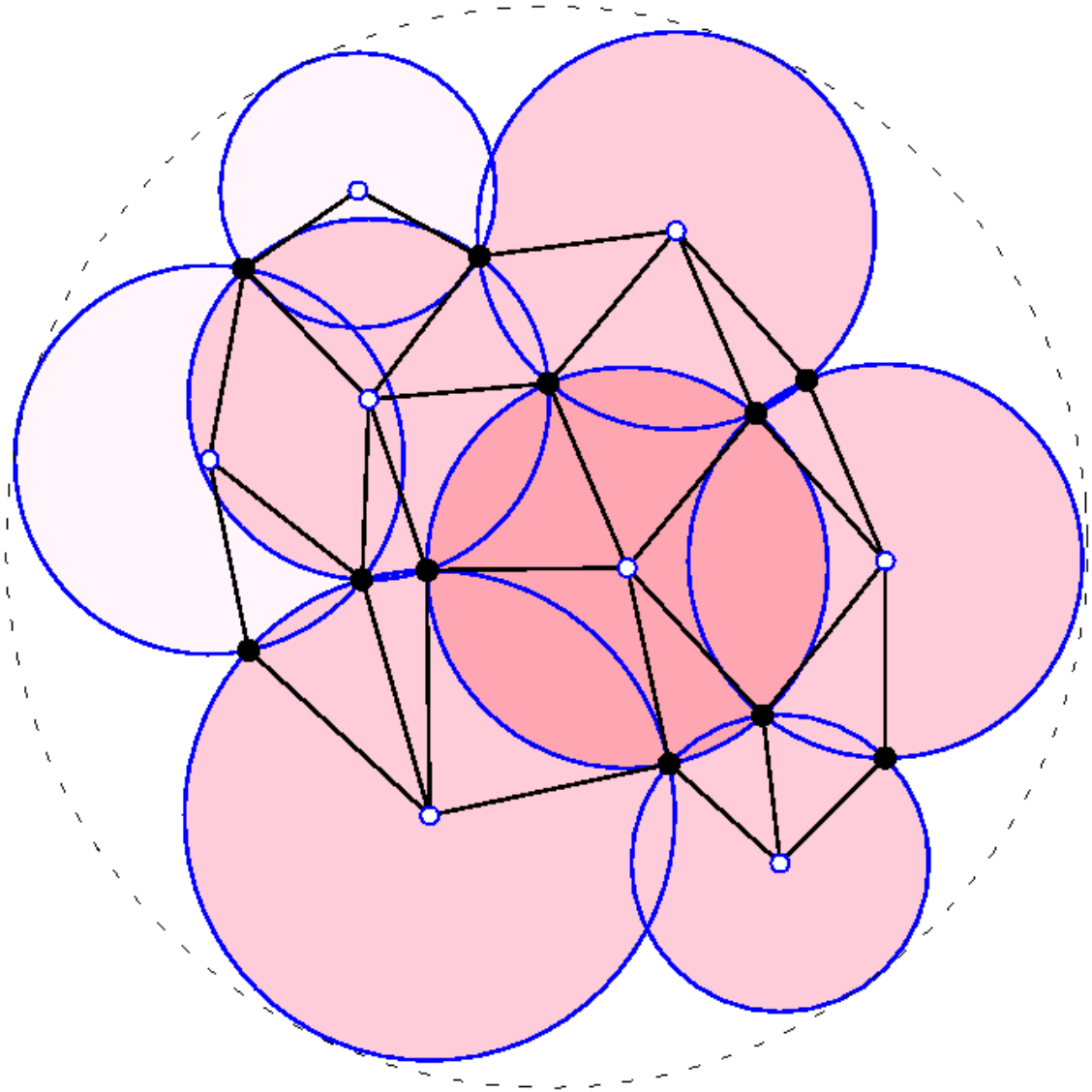


$$\varphi_1 = \frac{1}{i} \log \frac{r_1 + r_2 e^{i\theta}}{r_1 + r_2 e^{-i\theta}}, \tag{8}$$

Circle flower equations (cross ratio system)

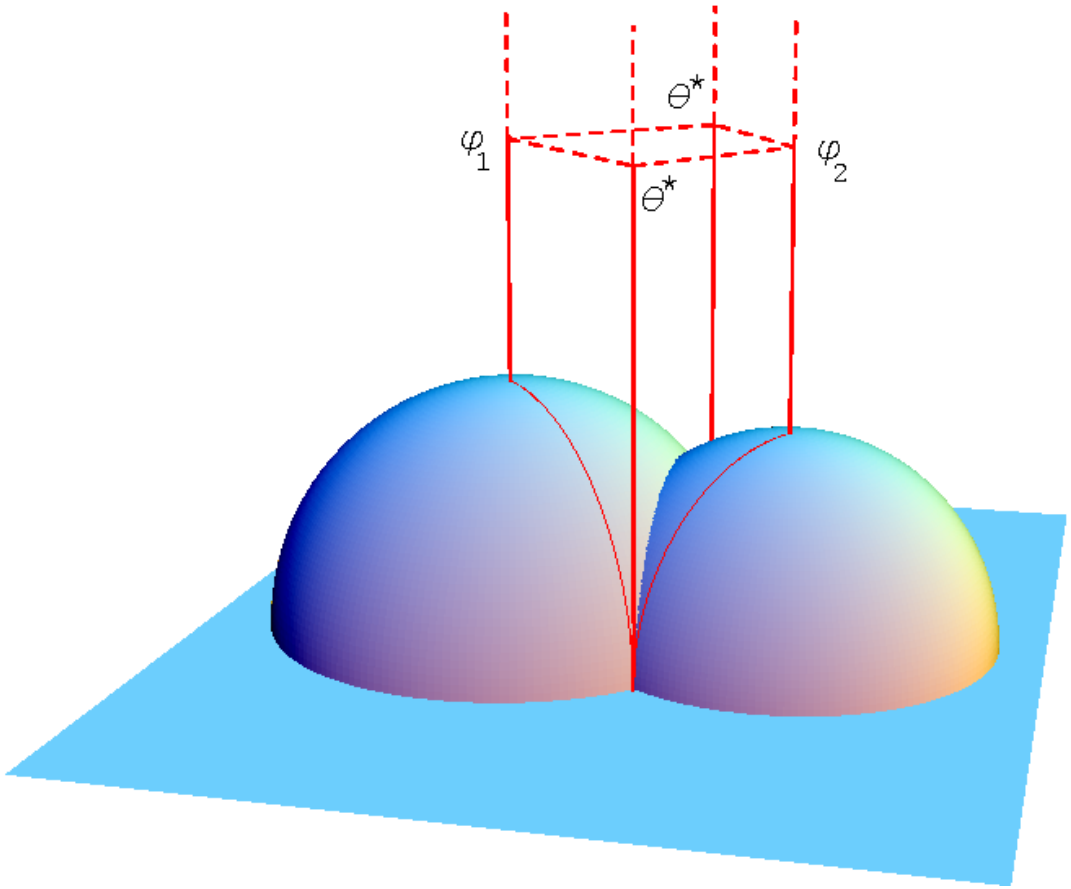
$$\sum_{(ij)\in star(i)} \varphi_{(ij)} = 2\pi, \qquad i \in V_{int}(\mathcal{G}) \ . \tag{9}$$

They are identical to the equation of motion in the Faddeev-Volkov model

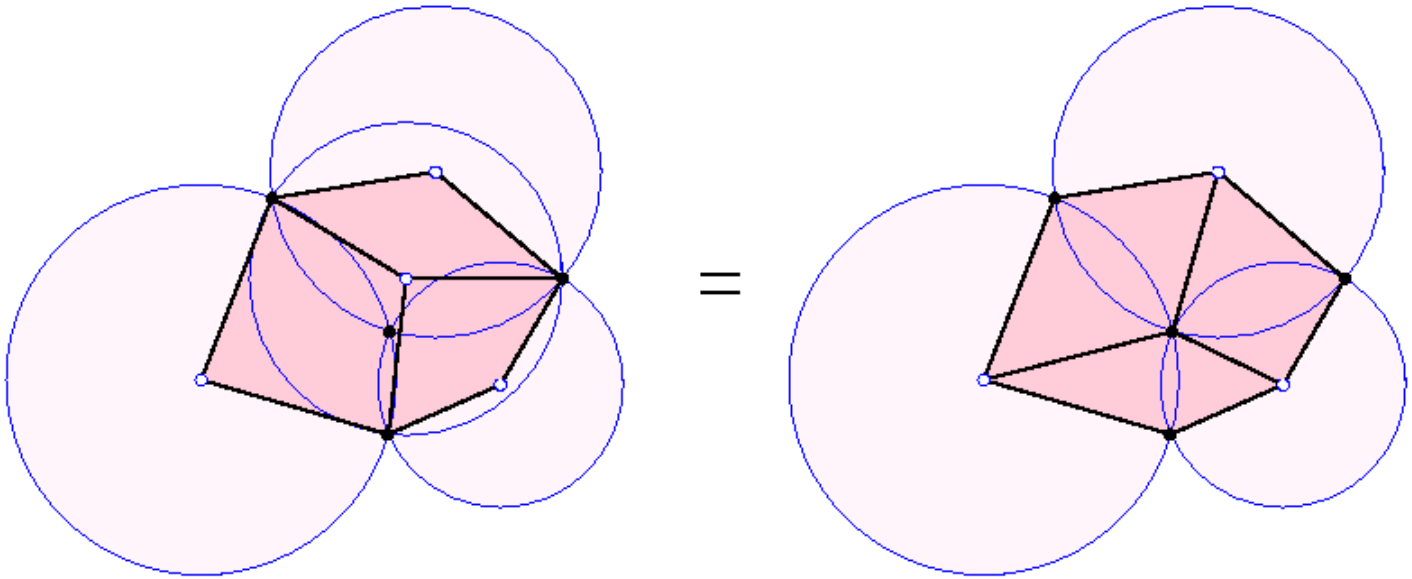


Star-triangle relation and hyperbolic geometry

Poincaré half-space model $\{x, y, z \in \mathbb{R} | z > 0\}$, $ds^2 = (dx^2 + dy^2 + dz^2)/z^2$.



Star-triangle-circle relation



$$V_{\bigstar} \;=\; 2\,\text{Л}(\frac{\theta_1}{2}) + 2\,\text{Л}(\frac{\theta_2}{2}) + 2\,\text{Л}(\frac{\theta_3}{2}) \;+ \frac{1}{2}\mathcal{A}_{\bigstar}[\rho_0^{(cl)}, \rho_1, \rho_2, \rho_3] + \text{boundary term} \qquad (10)$$

$$V_{\triangle} \;=\; 2\,\text{Л}(\frac{\pi-\theta_1}{2}) + 2\,\text{Л}(\frac{\pi-\theta_2}{2}) + 2\,\text{Л}(\frac{\pi-\theta_3}{2}) + \frac{1}{2}\mathcal{A}_{\triangle}[\rho_1, \rho_2, \rho_3] + \text{boundary term} \qquad (11)$$

$$V_{\text{tetrahedron}} = V_{\bigstar} - V_{\triangle} = \text{Л}(\theta_1) + \text{Л}(\theta_2) + \text{Л}(\theta_3)\,, \qquad (12)$$

$$2\,\text{Л}(\frac{\theta}{2}) - 2\,\text{Л}(\frac{\pi-\theta}{2}) = \text{Л}(\theta)\,. \qquad (13)$$