RIGIDITY OF POLYHEDRAL SURFACES

(VARIATIONAL PRINCIPLES ON TRIANGULATED SURFACES)

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Discrete Differential Geometry
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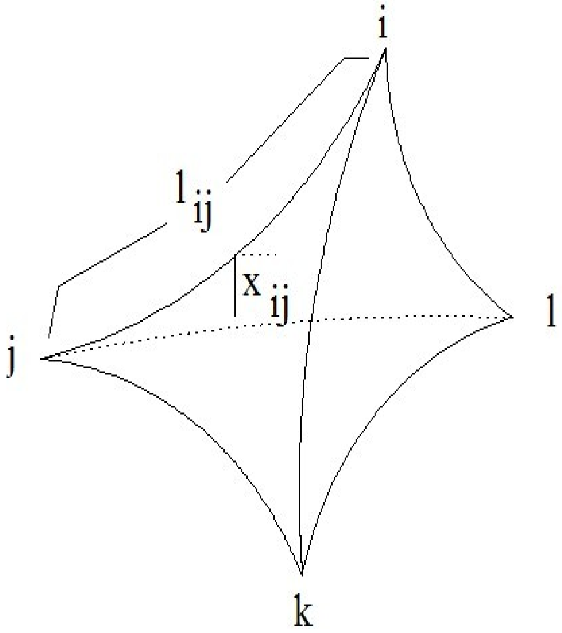
SCHLAEFLI FORMULA (1853)

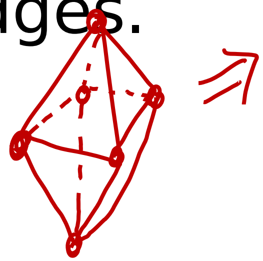
$$\partial V/\partial x_{ij} = -l_{ij}/2$$

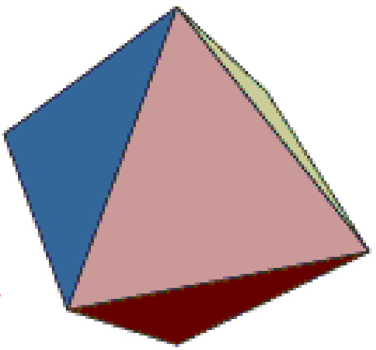


(1814-1895)

POLYHEDRAL SURFACES

Metric gluing of E^2 (or S^2, or H^2) triangles by isometries along edges.

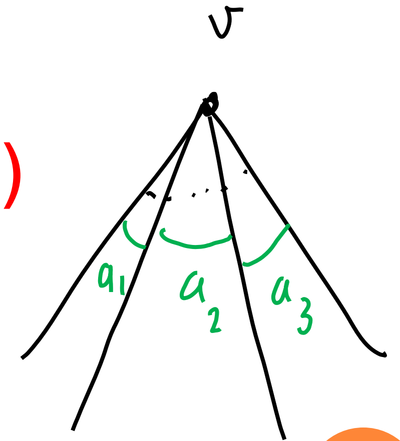




Metric: = edge lengths

Curvature k_0 at v:

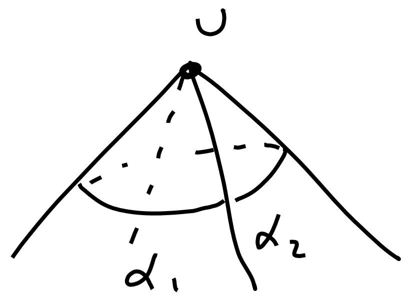
$$k_0\,(v) = 2\,\pi - (a_1 + a_2 + \ldots + a_m)$$



basic unit of curvature: *inner angle*

metric-curvature: determined by the *cosine law*

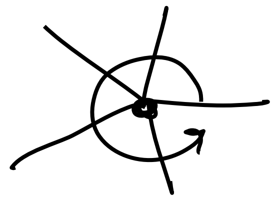
$K_0(v) > 0$



$\sum \alpha_i < 2\pi$

"convex"

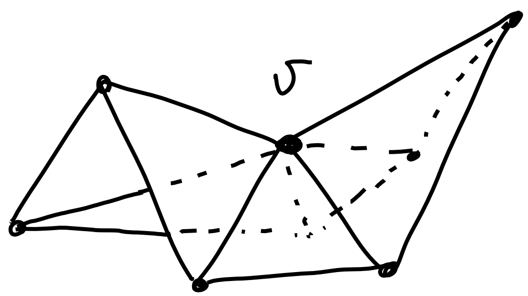
$K_0(v) = 0$



$\sum \alpha_i = 2\pi$

"flat"

$K_0(v) < 0$



$\sum \alpha_i > 2\pi$

"saddle"

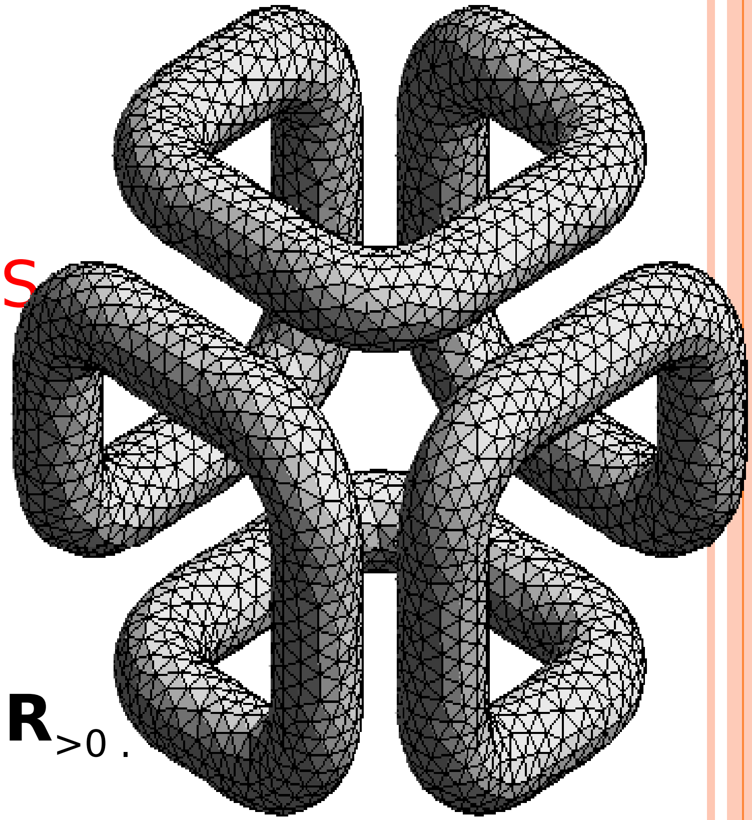
POLYHEDRAL METRIC

S = surface

T= triangulation of S

V= vertices in T

E= edges in T



polyhedral metric: $l : E \to \mathbf{R}_{>0}$.

discrete curvature: $k_0 : V \to R$

The relationship between metric l and curvature.

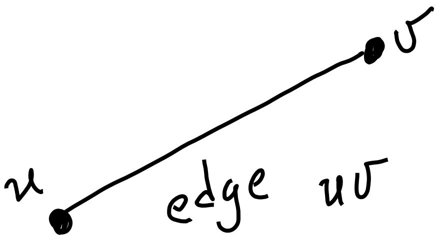
THURSTON'S WORK

A polyhedral metric on (S, T) is circle packing metric iff

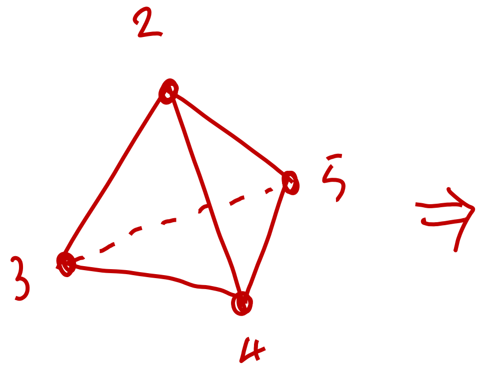
$r: V \to R_{>0}$

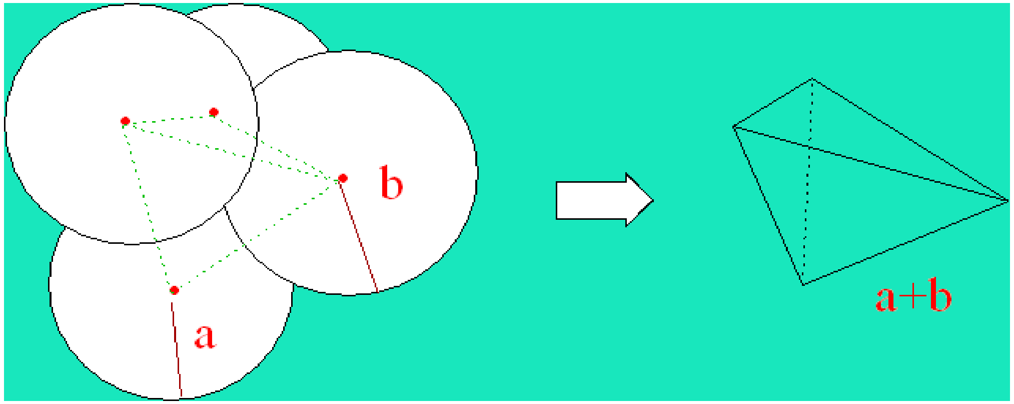
s. t., edge length

$$L(uv) = r(u) + r(v)$$



Eg. tetrahedron of circle packing type





THURSTON-ANDREEV RIGIDITY THM

If (S,T) closed triangulated,

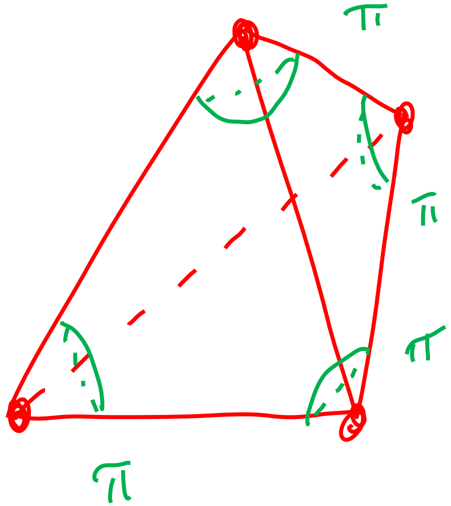
(a) A E^2 circle packing metric on (S,T) is determined by its k_0 curvature up to scaling.

(b) A H^2 circle packing metric on (S, T) is determined by its k_0 curvature.

Furthermore, the set $\{k_0\}$ is a convex polytope.

For a circle packing tetrahedron in $\mathbf{R}^3$, if all cone angles are π, then

Thurston Andreev say it is regular.

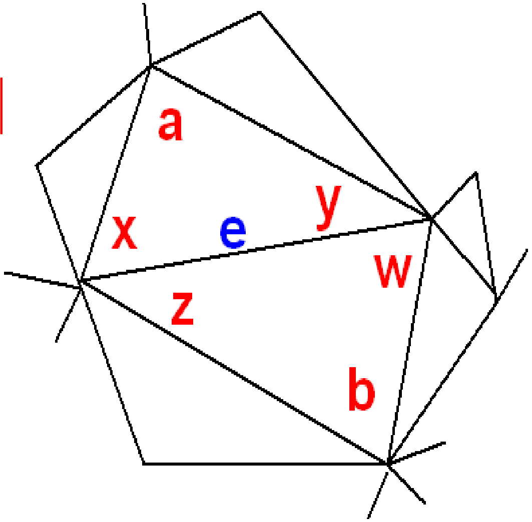


$\Rightarrow$ regular !

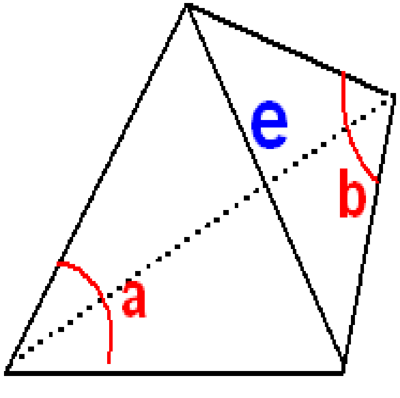
Rivin's Rigidity thm (Ann. Math, 1994)

A E^2 polyhedral metric on (S,T) is determined up to scaling by the φ_0 curvature,

φ_0: E → **R** sending e to π-a-b



If all $a+b = 2\pi/3$

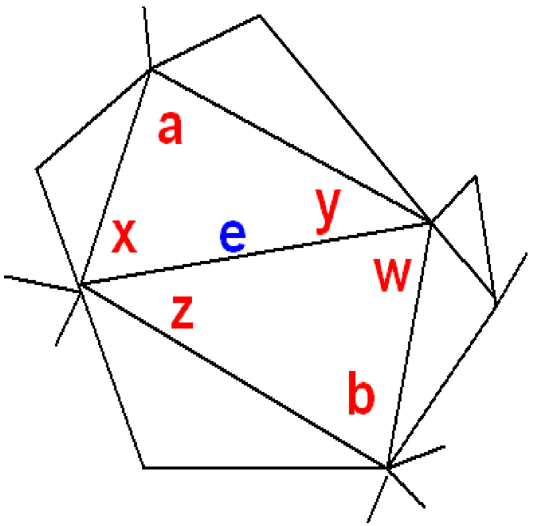


then regular tetrahedron.

LEIBON'S RIGIDITY THEOREM (GEO. & TOP., 2002)

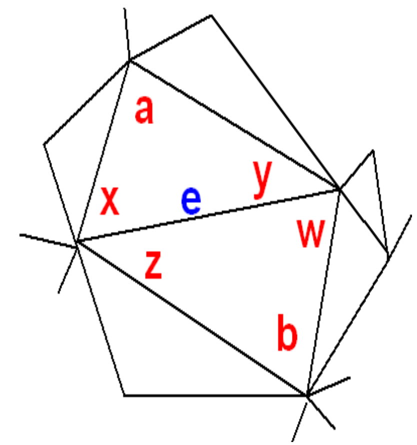
A H^2 polyhedral metric on (S, T) is determined by the ψ_0 curvature:

ψ_0 : E → **R** sending e to (x+y+z+w-a-b)/2.



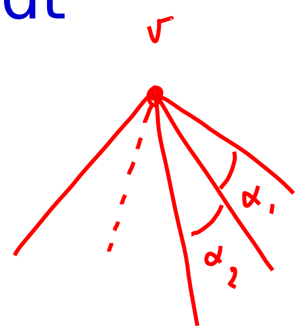
NEW CURVATURES

Let h ε **R**. Given a E^2, or S^2, or H^2 polyhedral metric on (S, T), define k_h, ψ_h, φ_h as follows:



$$\varphi_h(e) = \int_a^{\pi/2} \sin^h(t)\,dt + \int_b^{\pi/2} \sin^h(t)\,dt$$

$$\psi_h(e) = \int_0^{(x+y-a)/2} \cos^h(t)\,dt + \int_0^{(z+w-b)/2} \cos^h(t)\,dt$$



$$k_h(v) = (4-m)\pi/2 - \Sigma_a \int_{\pi/2}^{a} \tan^h(t/2)\,dt$$

where *a*'s are angles at the vertex v of degree m.

POSITIVE CURVATURE

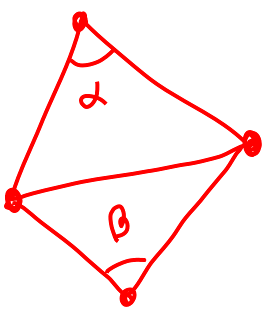
Positive curvature condition is independent of h, i.e.,

$$\varphi_h(e) \geq 0 \quad (\text{or} \quad \psi_h(e) \geq 0)$$

iff

$$\varphi_0(e) \geq 0 \quad (\text{or} \quad \psi_0(e) \geq 0),$$

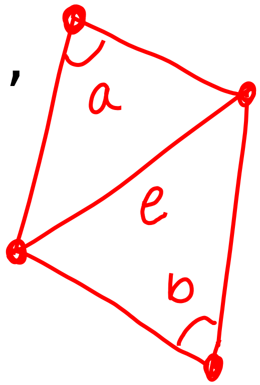
which is the Delaunay condition:



$\alpha + \beta \leq \pi \iff \phi_h(e) \geq 0$

Example,

$$\varphi_{-1}(e) = \ln(\tan(a)) + \ln(\tan(b)),$$



$$\varphi_{-2}(e) = \cot(a) + \cot(b)$$

appeared in the finite element approximation of the discrete Laplacian operator (Bobenko-Springborn, et al.)

$$\Delta(f)(v) = \sum_u \varphi_{-2}(uv)\,(f(u) - f(v)).$$

Thm 1. For any real h and any (S, T),

(i) A E^2 circle packing metric on (S, T) is determined up to scaling by k_h curvature.

(ii) A H^2 circle packing metric on (S, T) is determined by k_h curvature.

(iii) If $h \leqq -1$, an E^2 polyhedral metric on (S, T) is determined up to scaling by φ_h curvature.

(iv) If $h \leqq -1$ or ≥ 0, a S^2 polyhedral metric on (S, T) is determined by φ_h curvature.

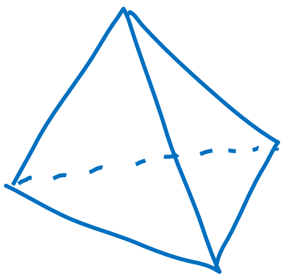
(v) If $h \leqq -1$ or ≥ 0, a H^2 polyhedral metric on (S, T) is determined by ψ_h curvature.

This theorem should be true for all h.

HYPERBOLIC METRIC ON SURFACE W/ BOUNDARY

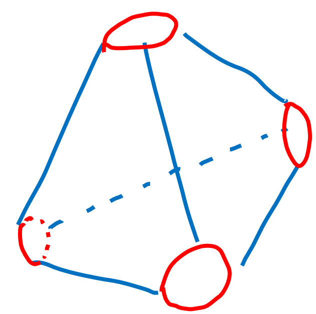
H^2 polyhedral metrics on closed trianguled surfaces

(S,T)



truncated triangulation

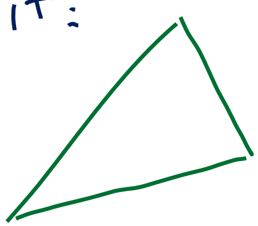
$(\tilde{S},\tilde{T})$



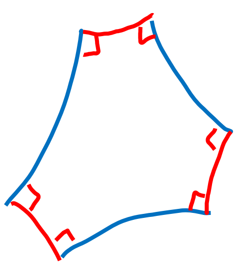
ideal triangulation

$E = \tilde{E}$

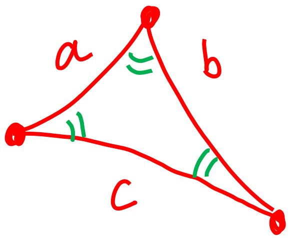
Basic unit:



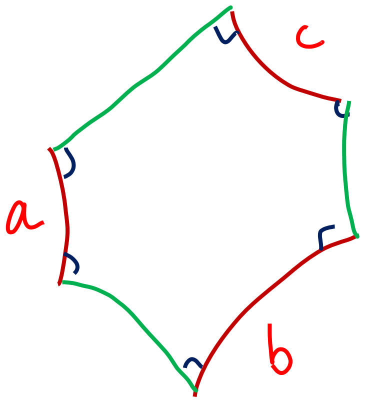
hexagon



HYPERBOLIC HEXAGONS



$a+b>c$ etc

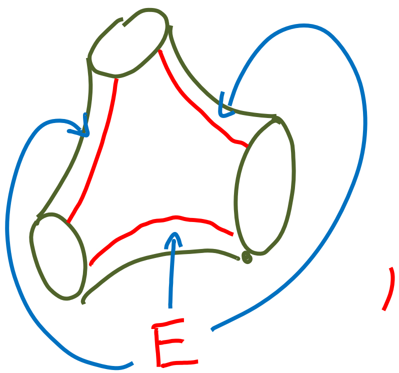


Fenchel-Nielsen: $\forall$ a,b,c >0, $\exists!$ right-angled hyperbolic hexagon with non-adjacent edge lengths a,b,c.

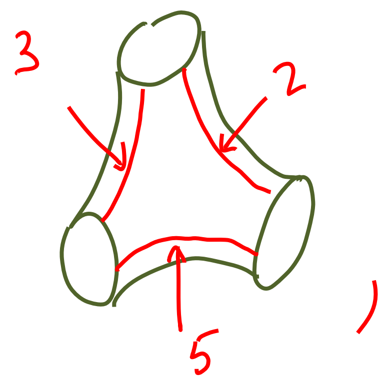
LENGTH COORD. OF T(S), S WITH BOUNDARY

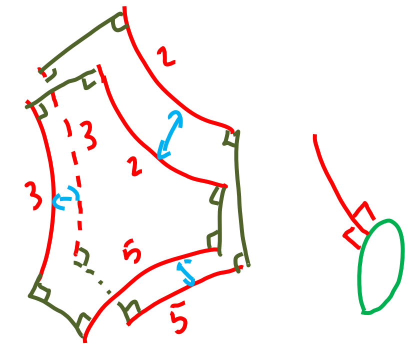
T(S) ={hyperbolic metrics d on S }/ isometry ≈ id.

Fix (S, T), each d in T(S) is constructed as follows.



$\ell : E \to \mathbb{R}_{>0}$,





isometric gluing

$\Rightarrow$ hyperbolic metric d_ℓ on S. These are all metrics in T(S).

THE LENGTH COORDINATE

This shows that: for an ideal triangulated surface

1. The Teichmuller space T(S) can be parameterized by $R^E_{>0}$ using the length $l: E \to R_{>0}$.

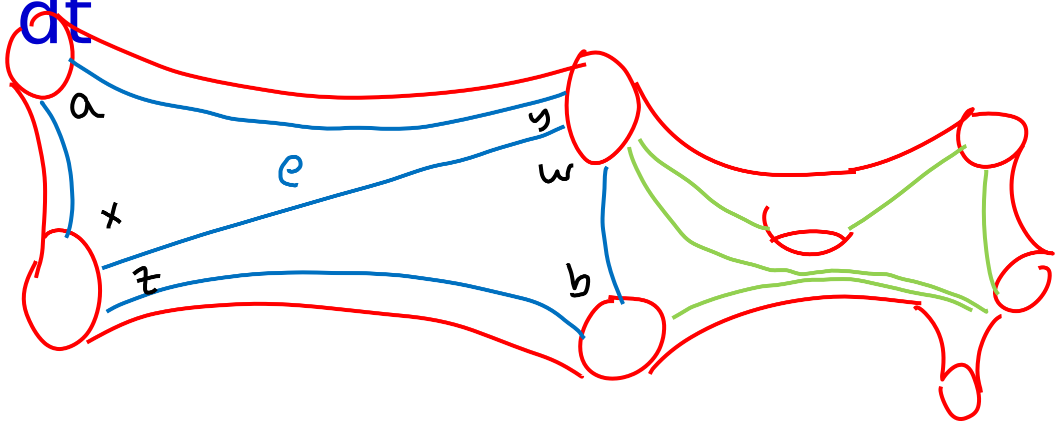
2. The Teichmuller space is simpler than the space of all polyhedral metrics on a closed triangulated surface (X, T).

Over the past 80 years, analysts, geometers and topologists have proved many fantastic theorems about T(S). Now it is probably the time to establish their counterparts for polyhedral metrics.

THE CURVATURE COORD.

For hyperbolic metric l: E -> $R_{>0}$, and h in **R** define

$$\psi_h(e) = \int_0^{(x+y-a)/2} \cosh^h(t)\,dt + \int_0^{(z+w-b)/2} \cosh^h(t)\,dt$$



$$\Psi_h : T(S) \longrightarrow \mathbb{R}^E$$

Define

$$d \longmapsto \Psi_h$$

Thm 2. For any h in **R**, any (S,T), the map

$$\Psi_h: \; T(S) \to \mathbf{R}^E$$

is a smooth embedding.

Furthermore, if $h \geq 0$, then the image $\Psi_h(T(S))$ is an open convex polytope so that $\Psi_h(T(S)) = \Psi_0(T(S))$.

Thm (Guo). If $h < 0$, the images $\Psi_h(T(S))$ are open convex polytopes.

Together with the works of Ushijima, Bowditch-Epstein, Hazel, Kojima on Delaunay decomposition (= “positive Ψ_h curvature”), we have,

Corollary. For a surface S w/ boundary, there exists a family of self-homeomorphims of the moduli space of curves preserving the natural cell structure.

VARIATIONAL PRINCIPLE

Thurston and Andreev’s proofs were excellent but not variational.

The first proof using variational principle was given by Colin de Verdiere in 1991 (Inv. Math.). (Bobenko-Springborn, et al).

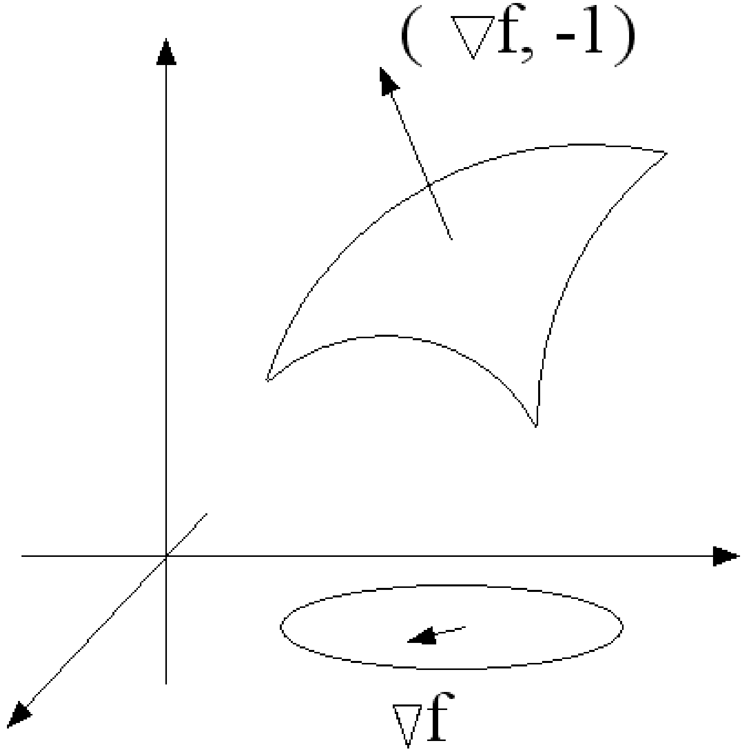
The idea is to construct an energy E(r) of a circle packing metric r, s.t.,

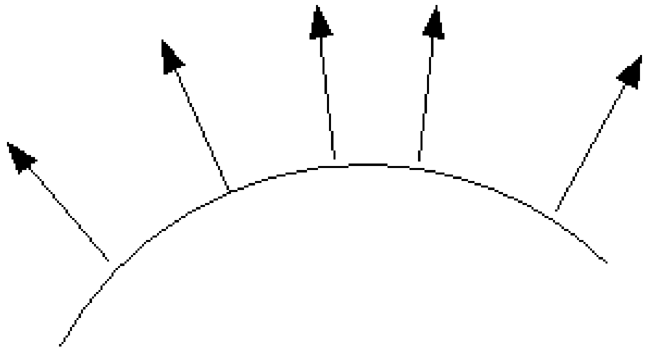
(i) its gradient is the curvature k_0 of r

(ii) E(r) is strictly convex.

BASIC LEMMA. If f: U→ R is smooth strictly convex/concave and U is an open convex set in R^n, then ∇f: U → R^n is injective.

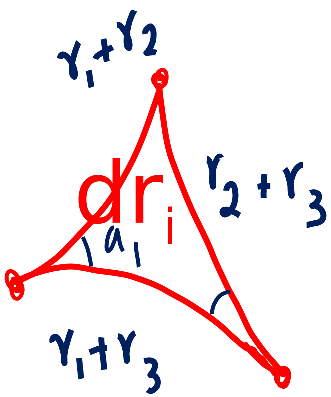
PROOF.

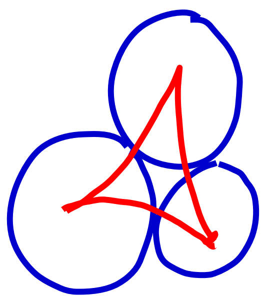




COLIN DE VERDIERE'S ENERGY

For a H^2 triangle of edge lengths $r_1 + r_2$, $r_2 + r_3$, $r_3 + r_1$ and inner angles a_1, a_2, a_3 , the 1-form





$$w = \sum a_i / \sinh(r_i)$$

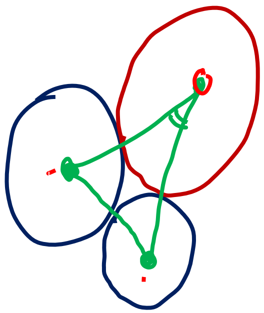
$$= \sum a_i \, du_i$$

is closed.

Its integration $F(u) = \int^u w$ is strictly concave in u.

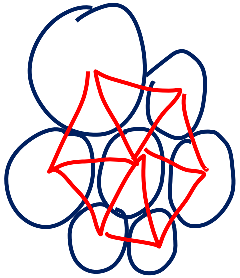
By the construction

$$\frac{\partial F}{\partial u_i} = a_i$$



For a circle packing metric r on (S, T),

define the energy W of it



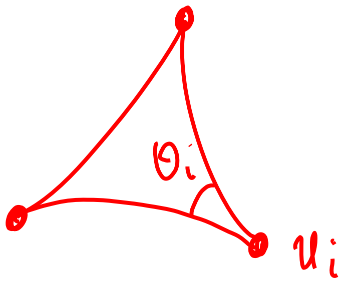
$$w(u) = \sum_{\Delta} F(u_i, u_j, u_k)$$

Then W concave, $\frac{\partial w}{\partial u_i} = 2\pi - K_0(u_i) \Rightarrow$

Thurston Andreev

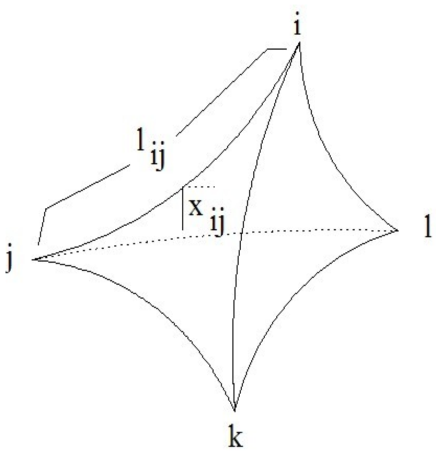
SCHLAEFLI FORMULA

Colin de Verdiere’s energy F should be considered as a 2-D Schlaefli formula:



$$\frac{\partial F}{\partial u_i} = \theta_i$$

Schläefli:



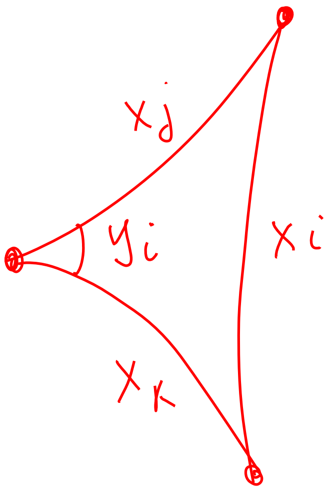
$$V = vol(X)$$

$$\frac{\partial V}{\partial x_{ij}} = \frac{1}{2} l_{ij}$$

(S^3)

Question: find all functions F on the lengths x of a triangle (y being angles) so that for some functions f, g,

$$\partial F\ (x) / \partial\ f(x_i) = g(y_i)$$



for indices i=1,2,3, i.e., the 1-form

$$w = \Sigma\ f(y_i)\ d\ g(x_i)$$

is closed.

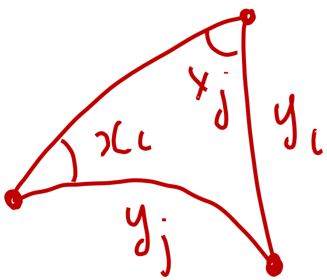
This is the same as finding all 2-D Schlaefli type identities.

There is a similar question for radius parameters.

THE COSINE LAW

The cosine law is

$$\cos(\sqrt{\lambda}\, y_i) = (\cos x_i + \cos x_j \cos x_k)/(\sin x_j \sin x_k)$$



triangle in $\mathbb{E}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$ curvature λ

Consider the cosine law function y=y(x)

$\cos(y_i) = [\cos(x_i) + \cos(x_j) \cos(x_k)]/[\sin(x_j) \sin(x_k)]$

where x, y are in $\mathbf{C}^3$, {i, j, k}={1,2,3}, x=(x_1, x_2, x_3) etc.

Thm 3. For the cosine law function $y=y(x)$, all closed 1-forms of the form

$$w = \Sigma\, f(y_i)\, d\, g(x_i)$$

are, up to scaling and complex conjugation,

$$w=\Sigma_i \left(\int^{y_i} \sin^h(t)\, dt \;/ \sin^{h+1}(x_i)\right) dx_i$$

i.e., $f(s) = \int^s \sin^h(t)\, dt$,

$g(s) = \int^s \sin^{-h-1}(t)\, dt.$

All w's are holomorphic.

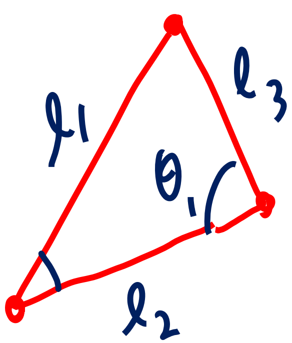
There is a similar result $y=y(r)$ if $r_i = 1/2(x_j + x_k - x_i)$.

SKETCH OF PROOF

- That the 1-forms are closed, direct check.

- These are the complete list of all forms :

Uniqueness of Sine Law. If f,g non-constant functions s.t.,



$$\frac{f(\theta_i)}{g(l_i)} \text{ independent of } i$$

$$\Rightarrow \quad f(t) = \sin^h(t), \quad g(t) = t^h$$

.

By specializing theorem 3 to various cases of triangles in S^2, E^2, H^2, and hyperbolic hexagons,

we are able to find the complete list of all energy functions which are convex/concave and produce a proof of thm1, and the rigidity part of thm 2.

MODULI SPACES AND COORDINATE

The thms 1 ,2 show that φ_h, ψ_h, K_h can serve as coordinates for various moduli spaces of polyhedral metrics.

What are the images of the moduli spaces under these coordinates?

The basic result is Thurston-Andreev thm that the space of k_0 is a convex polytope for c.p. metrics.

IMAGES OF THE MODULI SPACES

Thm 4. Let $h \leq -1$ and (S, T) be a closed triangulated surface.

(a) The image Φ_h of all E^2 polyhedral metrics on (S,T) in

φ_h curvature is a proper codimension-1 smooth submanifold X in R^E.

(b) The image Ψ_h of all H^2 polyhedral metrics in ψ_h curvature is the intersection of an open convex polytope with a component of $R^E - X$ in R^E.

PROOF THURSTON-ANDREEV'S THEOREM (AFTER MARDEN-RODIN)

Let (S, T) be a closed triangulated surface with
V = the set of all vertices.

Let $R^V_{>0,1}$ be the space of all E^2 circle packing metrics so that the sum of all radii =1.

Let K: $R^V_{>0,1}$ -> R^V be the curvature map sending r to k_0.

Thurston-Andreev: $K(R^V_{>0,1})$ = a convex polytope.

Gauss-Bonnet : $K(R^V_{>0,1})$ in a hyper-plane P in R^V.

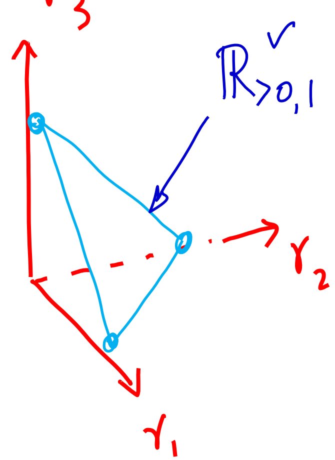
MARDEN-RODIN'S PROOF

Thurston-Andreev's rigidity says K is a smooth embedding K: $R^V_{>0,1}$ -> P.

To see the shape of K($R^V_{>0,1}$), it suffices to understand its boundary.

Take a sequence $r^{(m)} \in \mathbb{R}^V_{>0,1}$ s.t. $r^{(m)} \to r^{(\infty)} \in \partial(\mathbb{R}^V_{>0,1})$

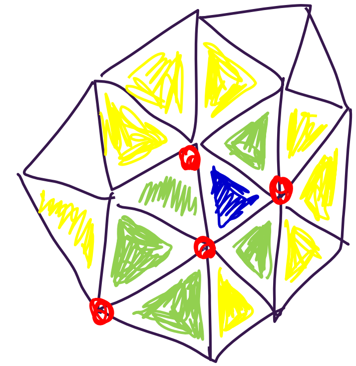
Thus, for some $v \in V$, $r^{(\infty)}(v) = 0$



Let $I = \{ v \in V \mid r^{(\infty)}(v) = 0 \}$

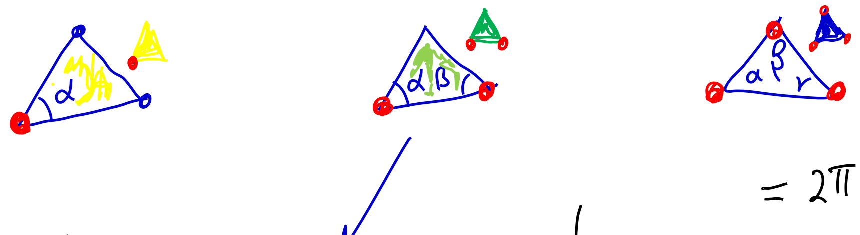
$I \neq \phi$ and $I \neq V$

Consider

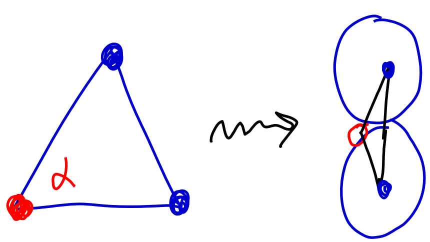


$$\sum_{v \in I} k_0(v) = 2\pi|I| - \sum_{\substack{\alpha \text{ at } v \\ v \in I}} \alpha$$

$$= 2\pi|I| - \sum \alpha - \sum \alpha+\beta - \sum \alpha+\beta+\gamma$$

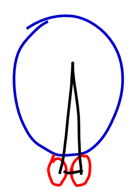


$= 2\pi|I| - \pi|F_I|.$



$\alpha = \pi$

$<$



$\alpha+\beta = \pi$

$<$

$\alpha+\beta+\gamma = \pi$

SUMMARY

- 2-D Schlaefli type formulas -> action functionals.
- Convexity of energy -> rigidity.
- Thurston's direct analysis of singularity formation -> the shape of the moduli spaces in curvature coordinates.

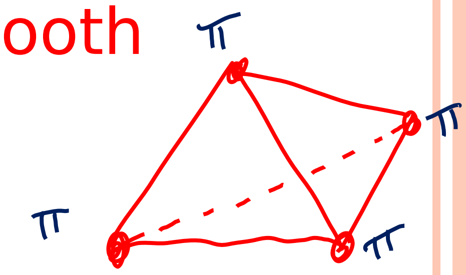
Question. Given a closed triangulated surface (S, T) and

f: V → **R**, is the space of all E^2 (or H^2) polyhedral metrics on (S, T) with $k_0 = f$ a cell ?

Supporting evidences:

(a) Teichmuller spaces,

(b) we have shown that the spaces are smooth manifolds.



Eg. Is the space of tetrahedra w/ cone angles π homeomorphic to the plane?

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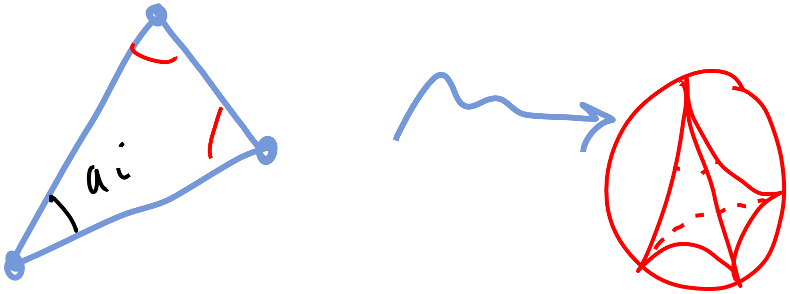
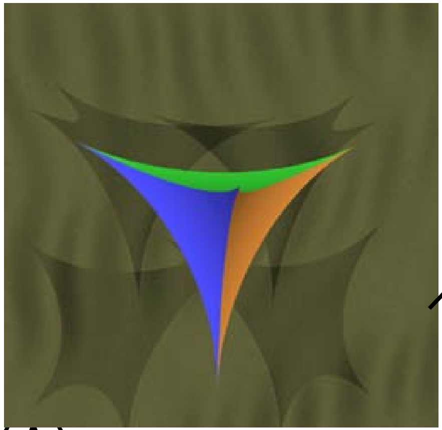
- Rivin, I., Ann. Math, 1994, vol 139.
- Leibon, G., Geo. & Top. 2002, vol 6.
- Marden, A, Rodin, B., Lecture Notes in Math, 1435.
- Luo, F., arXive: math.GT0612714.

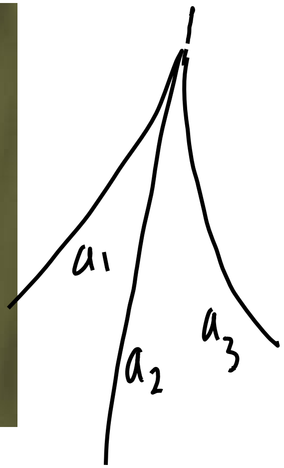
Thank you.

Thank you.

Rivin: E^2 triangle Δ -> ideal hyperbolic tetrahedron T(Δ) of the same angle.



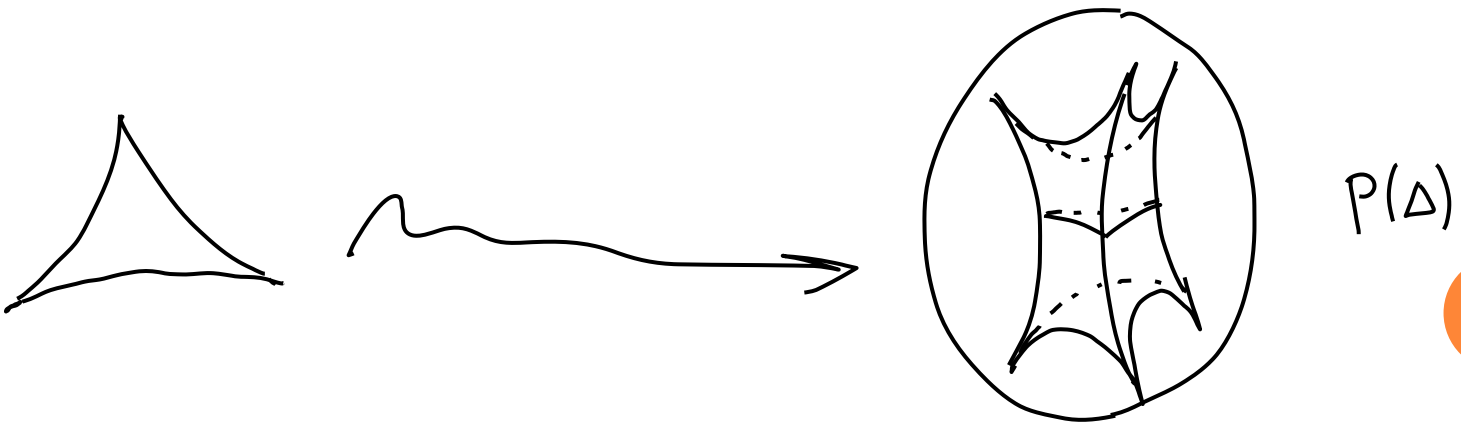




Rivin's energy: hyperbolic volume of T(Δ).

Leibon: H^2 triangle Δ -> ideal hyperbolic prizm P(Δ) of the same angle

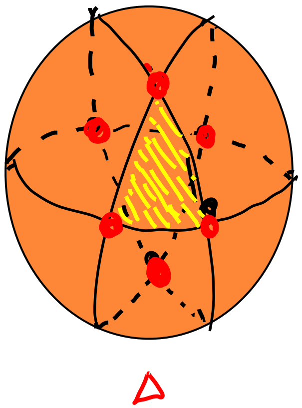
Leibon's energy: = hyperbolic volume of P(Δ).

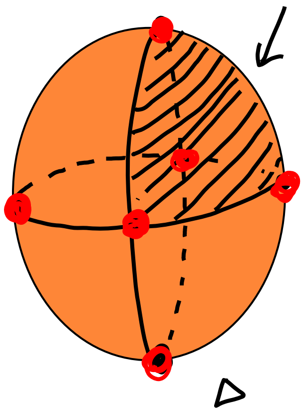


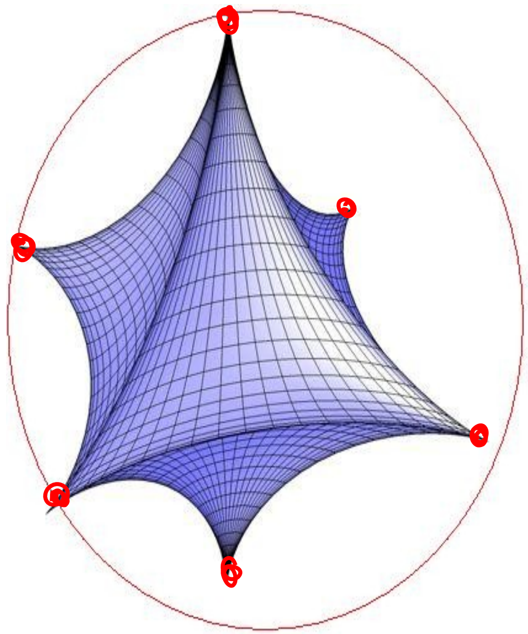
Energy of spherical triangle?

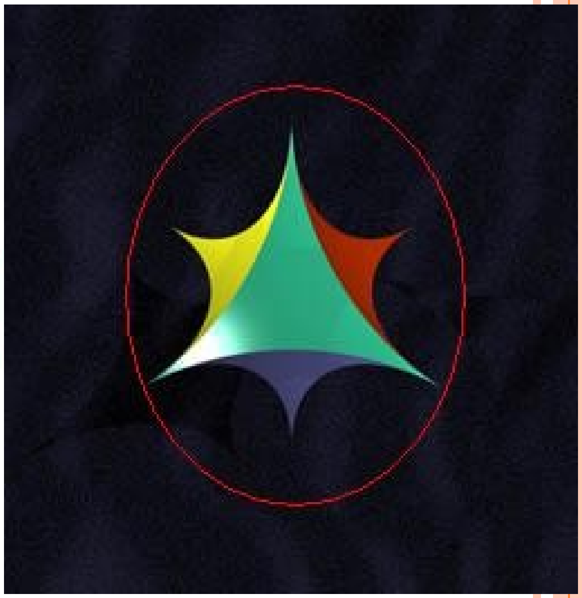
Spherical triangle Δ

A spherical triangle Δ is associated to a hyperbolic ideal octahedron $O(\Delta)$.





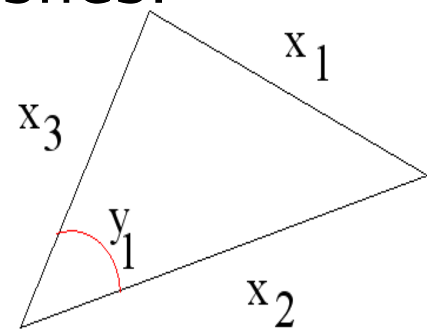




Energy=hyperbolic volume

The function F, by the construction, satisfies:

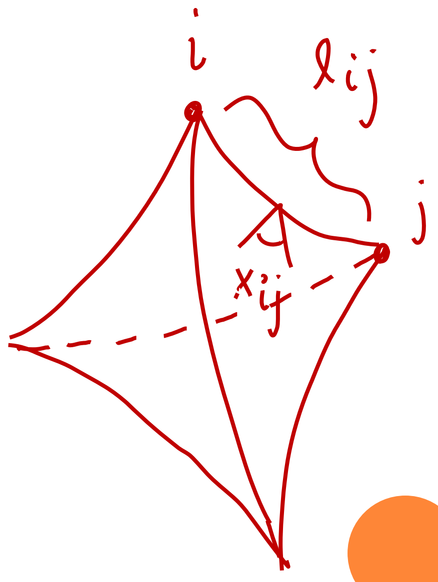
$$\partial F(x)/\partial x_i = \ln(\tan(y_i/2)).$$

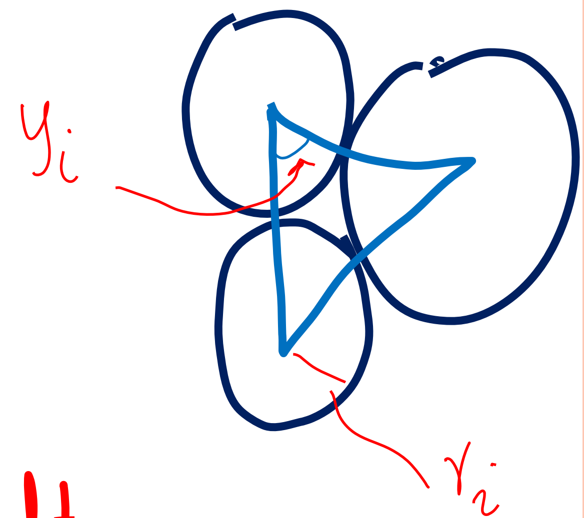


Let us call it the F-energy of the triangle.

Recall 3-dim Schlaefli formula:

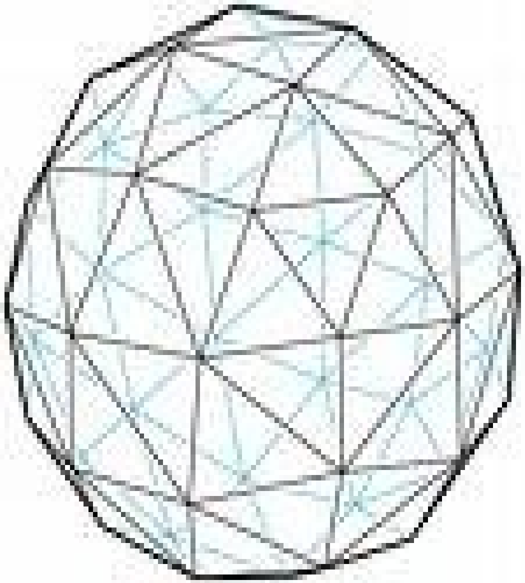
$$\frac{\partial V}{\partial x_{ij}} = -\frac{l_{ij}}{2}$$





$$\eta_q = \sum_{i=1}^{3} \frac{\int^{y_i} \tan^q\left(\frac{t}{2}\right) dt}{\cos^{q+1}(r_i)} \, dr_i$$

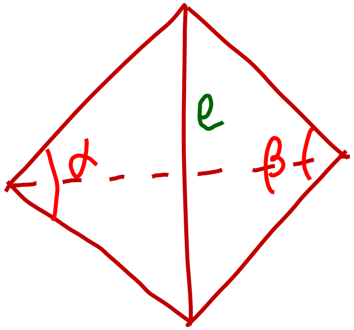
For a E^2 polyhedral metric L: E → **R** on (S, T), define the F-energy W of the metric L to be the sum of the F-energies of its triangles.



W is convex and the gradient,

$$\nabla W = \Phi_{-1}.$$

This is how we prove the thm 1 (iii) for Φ_{-1} curvature.

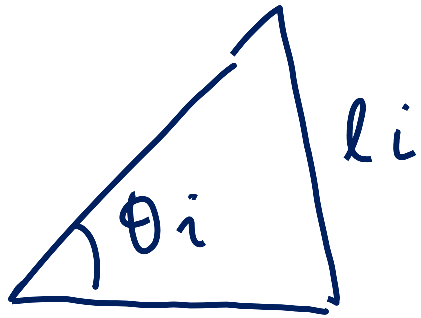


$$\Phi_{-1}(e) = \ln \tan\left(\frac{\alpha}{2}\right) + \ln \tan\left(\frac{\beta}{2}\right)$$

Eg. E^2 triangle of angles θ_i and edge length l_i, the 1-forms

$$\omega = \sum_i \int^{\theta_i} \sin^q(t)\,dt \Big/ l_i^{q+1}\, dl_i$$

are closed. For q=0, the form is:

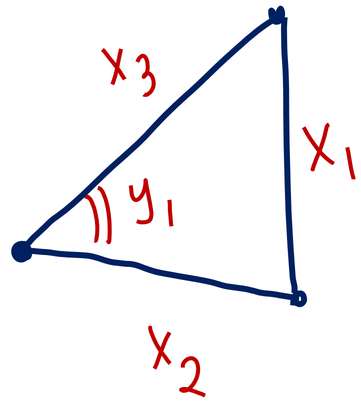


$$\omega_0 = \sum_i \frac{\theta_i}{l_i}\, dl_i$$

Its integration $\int^l \omega_0$ was first found by Cohn, Kenyon, Propp as a *partition function* of the dimer model in 2001. Its Legendre transformation gives Rivin’s energy.

Eg. For a E^2 triangle of lengths x and angles y, 1-form w

$$w = \Sigma_i \ \ln(\tan(y_i/2))\, d\, x_i$$



is closed.

Integrating w and obtain a function of x,

$$F(x) = \int^x w$$

This $F(x)$ can be shown to be convex in x.

The Cosine Law

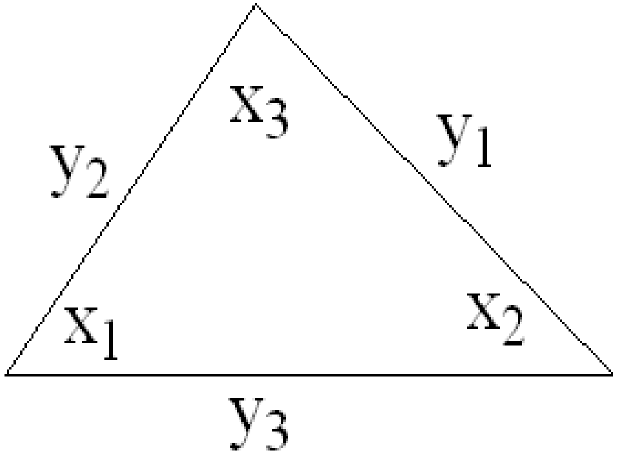
For a hyperbolic, spherical or Euclidean triangle of inner angles x_1, x_2, x_3

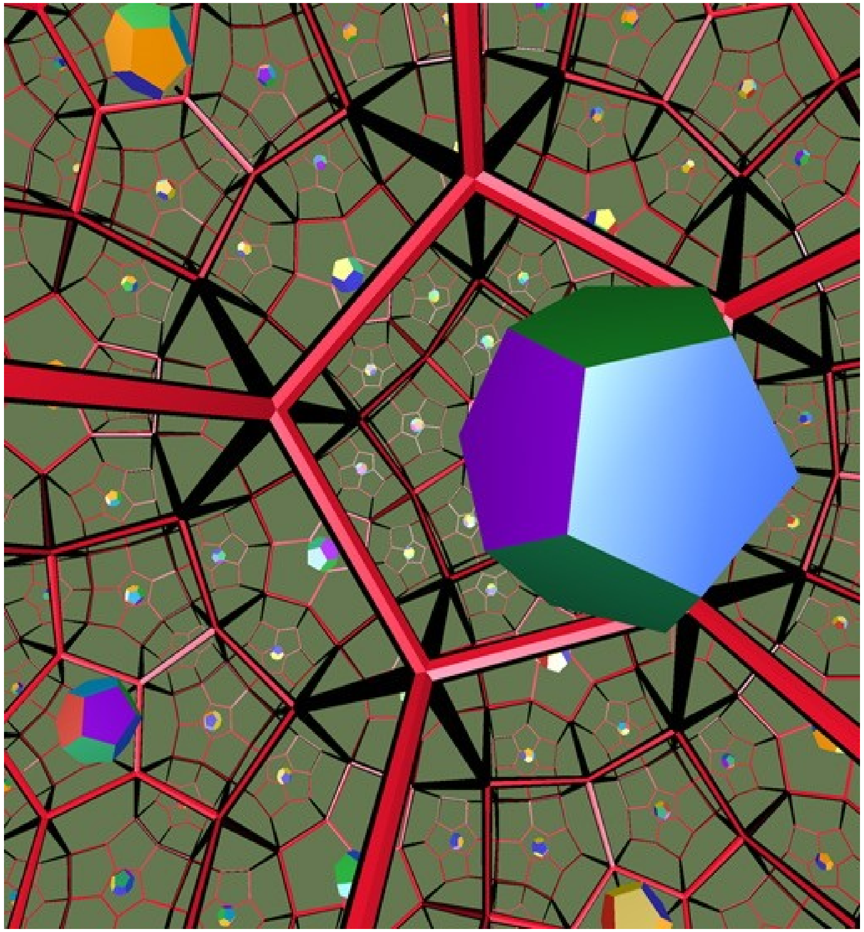
and edge lengths y_1, y_2, y_3 ,

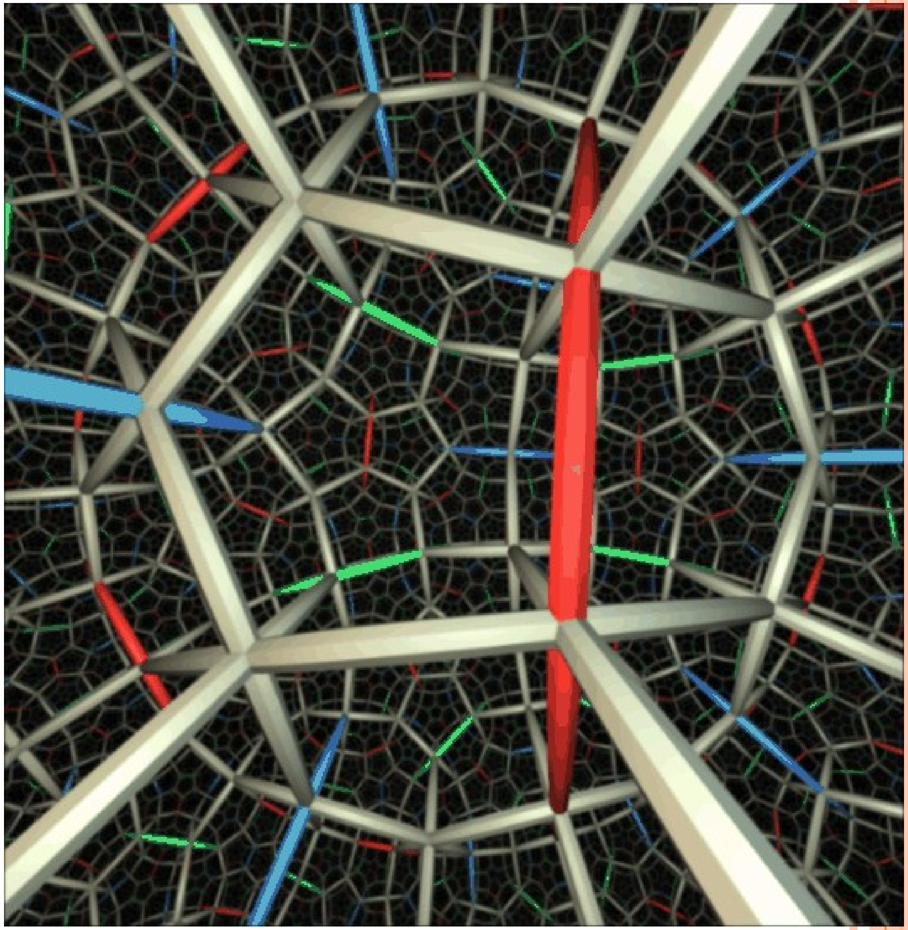
(S) $$\cos(y_i) = (\cos x_i + \cos x_j \cos x_k)/(\sin x_j \sin x_k)$$

(H) $$\cosh(y_i) = (\cos x_i + \cos x_j \cos x_k)/(\sin x_j \sin x_k)$$

(E) $$1 = (\cos x_i + \cos x_j \cos x_k)/(\sin x_j \sin x_k)$$

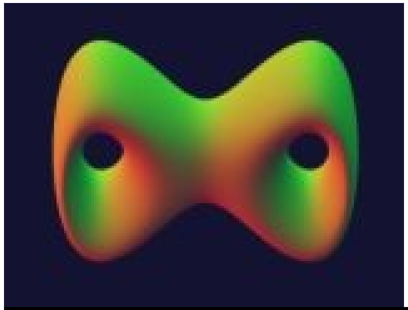




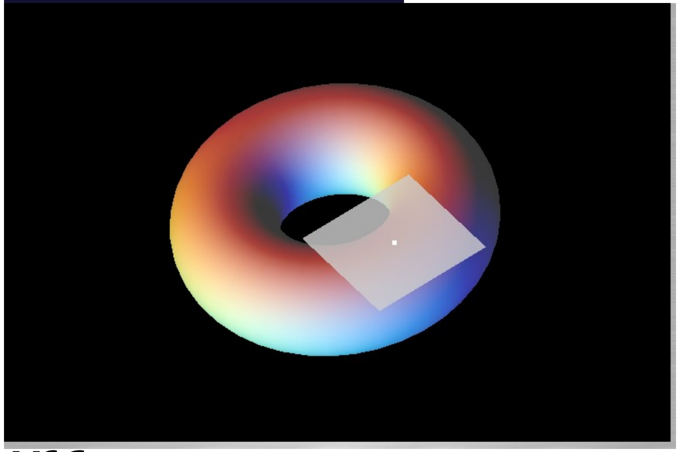


Thank you !

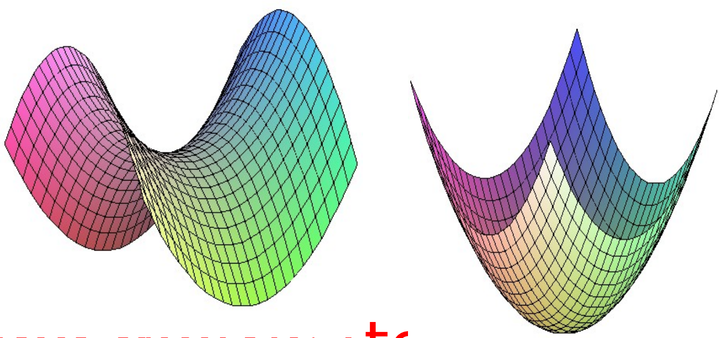
SMOOTH SURFACES



Metric = Riemannian metric



Curvature=Gaussian curvature



Basic question in surface geom

relationship between curvature and metric (tensor calculus)

Prop 2. For the cosine law function $y=y(x)$, the following holds,

$\sin(x_i)\sin(x_j)\sin(y_k) = A$ is independent of i,j,k (the sine law);

(ii) $\partial y_i/\partial x_i = \sin(x_i)/A$;

(iii) $\partial y_i/\partial x_j = \partial y_i/\partial x_i \cos(y_k)$. (Bianchi-identity?)

CELL-DECOMP. OF TEICHMULLER SPACE

Cor. 6. (Ushijima, Bowditch-Epstein, Hazel, Kojima, Guo, L,...)

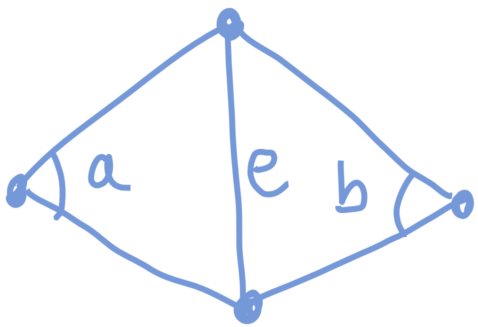
For $\forall \lambda \in \mathbb{R}$, and S cpt w/ boundary, the Teichmuller space T(S) has a cell-decomposition invariant under the action of the MCG,

$$T(S) = \bigcup_{\substack{T \text{ id.} \\ \text{triang}}} \Psi_\lambda^T(\mathbb{R}^E_{\geq 0})$$

Note

$$\phi_{-1}(e) = \int_{\pi/2}^{a} \frac{1}{\sin t}\,dt + \int_{\pi/2}^{b} \frac{1}{\sin t}\,dt$$

$$= \ln\tan\left(\tfrac{a}{2}\right) + \ln\tan\left(\tfrac{b}{2}\right)$$

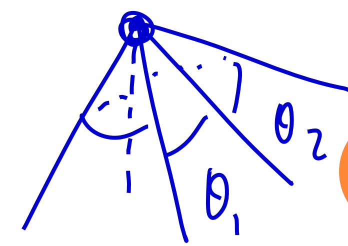


Given a circle packing metric $r: V \to \mathbb{R}_{>0}$, let $u_i = \int_{r_i}^{\infty} \frac{dt}{\sinh t}$, and $u = (u_1, \cdots, u_n) : V \to \mathbb{R}$.

Define its energy

$W(u)$ = sum of the energies of F over triangular faces.

$$\frac{\partial W}{\partial u_i} = \theta_1 + \cdots + \theta_n$$



$\square$