Modeling Flexible Multibody Systems by Moving Dirichlet Boundary Conditions

Robert Altmann

1 Institut für Mathematik, TU Berlin, Straße des 17. Juni 136, D–10623 Berlin Charlottenburg, Germany
raltmann@math.tu-berlin.de

Abstract

It is shown that coupled systems from flexible multibody dynamics can be described by the same model as a single body with unknown Dirichlet boundary conditions. Based on a model for moving Dirichlet conditions, the coupling surface between adjacent bodies may even change its position and size. Since the position of the constraint is assumed to be known a priori, the constraint remains linear.

The reduction of a flexible multibody system to the structure of a single body is shown by means of an example with two bodies. Thus, it is sufficient to analyse the single body case in terms of regularization and efficient simulation tools. The paper closes with an example motivated by the pantograph and catenary system.

Keywords: flexible multibody systems, Dirichlet boundary conditions, coupled system

1 Introduction

Constrained partial differential equations (PDEs) arise in automatic modeling of flexible multibody systems (MBSs) but also more generally in multiphysics systems. We consider multibody systems where the coupling between the bodies is given on the boundary, i.e., by Dirichlet boundary conditions. Since the constraints might depend on data which is not known a priori (e.g. movement of adjacent bodies), it is preferable to involve the Dirichlet conditions in form of a weak constraint [1].

The presented model allows to involve constraints on a moving part of the boundary. In contrast to contact problems, we assume here that the position of the constraint is known beforehand. This gives more flexibility but restricts the problem to linear constraints. With this generalization, the framework includes examples of the type pantograph and catenary [2], see Figure 1.

It is well-known that the semi-discretization in space of flexible MBSs leads to differential-algebraic equations (DAEs) of differentiation index 3. Within this concept, the index describes the needed smoothness of the inhomogeneity to guarantee a continuously differentiable solution [3, Ch. 3.3]. We write the constrained PDE in operator form, which leads to a so-called operator DAE that implements the structure of the system. Already in this continuous setting, i.e., before discretizing the operator DAE in space, the high index structure can be seen. Therefore, we analyse the operator DAE in terms of an inf-sup condition. Reducing the MBS to the single body case, we can reuse known index reduction techniques and simulation tools such as suitable preconditioners.

Figure 1. Illustration of a pantograph and catenary system as example of an application with moving Dirichlet boundary conditions.

The outline of this paper is as follows. In Section 2 we summerize the approach of including moving Dirichlet boundary conditions. In addition, we comment on index reduction techniques for the semi-discretized system which allows an efficient simulation of such systems.

The application to flexible MBSs in then part of Section 3. By means of a two body system, we show how to define the corresponding operators such that the coupled system has the same structure as a single body system with moving
Dirichlet boundary conditions. This includes the stability condition for the constructed constraint operator. An illustrative example, which is motivated by the pantograph and catenary system, is given before we conclude in Section 4.

2 Mathematical Model of Moving Dirichlet Boundary Conditions

In this section, we show how to include Dirichlet conditions on a moving boundary part for the dynamics of an elastic body. We follow [4] where a model with suitable ansatz spaces is presented. In this approach, the Dirichlet conditions are enforced via the Lagrangian method. The resulting formulation is especially suited but not restricted to flexible MBSs, since in that case the Dirichlet conditions are not known a priori due to the motion of adjacent bodies [1].

2.1 Setting and Transformation

Let $\Omega \subset \mathbb{R}^d$ be the domain of interest with Dirichlet boundary conditions on a time-dependent boundary strip $\Gamma_D(t) \subset \partial \Omega$. To avoid time-dependent ansatz spaces, we introduce a transformation which maps a fixed compact subset $I \subset \mathbb{R}^{d-1}$ onto the Dirichlet boundary $\Gamma_D(t)$. We denote this transformation, which is assumed to be bi-Lipschitz and onto, by

$$\Phi(t) : I \rightarrow \Gamma_D(t).$$

Clearly, this requires that the boundary $\partial \Omega$ itself is Lipschitz. The bi-Lipschitz property of $\Phi(t)$ implies the boundedness of the derivative of $\Phi$ almost everywhere in space. As usual, the ansatz space for the deformation is the Sobolev space $H^1(\Omega)$ in $d$ components. The involved Lagrange multiplier is defined as a functional on $I$. Thus, we define the spaces

$$\mathcal{V} = [H^1(\Omega)]^d, \quad \mathcal{Q}^* = [H^{1/2}(I)]^d.$$

Note that the space $\mathcal{Q}$ for the Lagrange multiplier is defined via its dual space $\mathcal{Q}^*$. It remains to formulate the Dirichlet boundary condition in operator form. For this, we define as in [4] the time-dependent operator $\mathcal{B}(t) : \mathcal{V} \rightarrow \mathcal{Q}^*$ via

$$\langle \mathcal{B}(t)u, q \rangle_{\mathcal{Q}^*, \mathcal{Q}} := \int_{\Gamma_\nu(t)} u \cdot (q \circ \Phi(t)^{-1}) \bigg| \det \mathcal{D}\Phi(t)^{-1} \bigg| \, dx.$$  

**Remark 1.** If the transformation $\Phi$ equals the identity, i.e., $\Gamma_D(t) = I$, then $\mathcal{B}$ coincides with the trace operator. In this case, it is well-known that $\mathcal{B}$ satisfies an inf-sup condition [5, Lem. 4.7]. But also in the general case, $\mathcal{B}(t)$ satisfies an inf-sup condition under certain conditions on the transformation $\Phi$. For details we refer to [4, Lem. 2.2].

2.2 Saddle Point Formulation

With the help of the dual operator of $\mathcal{B}$, we are in the position to state the overall problem in saddle point form. Let $\mathcal{M}, \mathcal{D}$, and $\mathcal{K}$ denote the operators corresponding to the material properties. For the dynamics of elastic media, $\mathcal{M}$ is positive definite and includes the density of the material, $\mathcal{D}$ is a damping operator, and $\mathcal{K}$ reflects the stiffness of the material. For details concerning the theory of elasticity, we refer to [6, Ch. 2]. The equation of motion then reads

$$\mathcal{M}\ddot{u}(t) + \mathcal{D}\dot{u}(t) + \mathcal{K}u(t) = \mathcal{F}(t)$$

with right-hand side $\mathcal{F}$, initial conditions, and the boundary constraint that the deformation $u$ equals a given function $u_D$ on $\Gamma_D(t)$. The existence of solutions for such hyperbolic differential equations is discussed in [7, Ch. V]. The Dirichlet boundary condition in weak form reads $\mathcal{B}(t)u(t) = \mathcal{G}(t) := \mathcal{B}(t)u_D(t)$. Written in saddle point form with the Lagrangian method [1], we obtain the system

$$\begin{align*}
\mathcal{M}\ddot{u}(t) + \mathcal{D}\dot{u}(t) + \mathcal{K}u(t) + \mathcal{B}^*(\lambda(t)) &= \mathcal{F}(t) \quad \text{in } \mathcal{V}^*,
\mathcal{B}(t)u(t) &= \mathcal{G}(t) \quad \text{in } \mathcal{Q}^*
\end{align*}$$

for a.e. $t \in (t_0, T)$. The corresponding (consistent) initial conditions read

$$\begin{align*}
u(t_0) &= a_0 \in \mathcal{V}, \\
\dot{u}(t_0) &= a_1 \in [L^2(\Omega)]^d.
\end{align*}$$

Note that the time derivatives in this formulation should be understood in a weak sense. For further details on weak time derivatives and on used embeddings which show the well-posedness of the initial condition $a_1 \in [L^2(\Omega)]^d$, we refer to [8].

With system (1) we have found a suitable formulation for the inclusion of moving Dirichlet boundary conditions. In Section 3, we show that flexible MBSs, coupled by Dirichlet boundary conditions, lead to systems of the same structure. For this reason, we analyse the semi-discrete version of system (1) in terms of stability and possible index reduction techniques.
2.3 Semi-Discrete Equations

Using a method of lines [9, Ch. 3.4], we discretize system (1) in space first. Standard finite element schemes with \( q(t) \in \mathbb{R}^n \) as the coefficient vector of the approximation of the deformation and \( \mu(t) \in \mathbb{R}^m \) corresponding to the approximation of the Lagrange multiplier lead to a DAE of the form

\[
M \ddot{q} + D(\dot{q}) + K(q) + B(t)\mu = f(t),
\]

\[
B(t)q = g(t).
\]

Therein, the mass matrix \( M \) is positive definite and \( D, K \) denote the (nonlinear) discrete versions of the operators \( D \) and \( K \), respectively. It is well-known that systems of this type have index 3 if the matrix \( B \) is of full row rank [10]. Although the full rank property suffices for the solvability, we should also demand stability, i.e., \( B \) should satisfy in addition a discrete inf-sup condition independent of the used mesh-size. An example of a stable discretization scheme, which involves edge-bubble functions at the boundary, is given in [4].

There are several approaches to tackle high index problems such as a differentiation of the constrains with stabilization [11] or using derivative arrays with additional projections [12, Ch. 4.6.2]. For more details on index reduction methods, we refer to [13, Ch. VII.2] and [3, Ch. 6].

Because of the given saddle point structure, it is possible to use the index reduction technique of minimal extension [14]. Therein, the derivatives of the constraint are added to the system and a minimal number of dummy variables is introduced such that the system is square. Note that a variant of this technique might already be applied to the operator DAE (1), cf. [15]. This approach allows to employ the Rothe method, i.e., to discretize in time first, without involving high index effects.

3 Application to Flexible MBS

To keep things clear, we only consider two coupled bodies in two space dimensions with a time-dependent coupling condition. Nevertheless, the technique works in the same manner for multiple bodies.

3.1 Coupling of Two Bodies

We consider the coupling of two domains with two types of boundary conditions, as shown in Figure 2. First, we have (standard) Dirichlet boundary conditions on the time-independent part \( \Gamma_1 \subset \partial \Omega_1 \). Secondly, we have a coupling condition which states that the boundaries \( \Gamma_{12}(t) \subset \partial \Omega_1 \) and \( \Gamma_{21}(t) \subset \partial \Omega_2 \) are glued together and thus, also have equal deformation. This constraint is modeled as a Dirichlet boundary condition on a moving boundary part in the sense of the previous section.

\[\begin{aligned}
\Omega_1 & \quad \Gamma_{12}(t) \\
\Omega_2 & \quad \Gamma_{21}(t)
\end{aligned}\]

Let \( u_1 \) and \( u_2 \) denote the deformation of the domains \( \Omega_1 \) and \( \Omega_2 \), respectively. The Sobolev spaces for the two domains are denoted by \( V_1 \) and \( V_2 \). Since we only need a transformation for the boundary \( \Gamma_{12} \), we define the spaces

\[
Q^*_{12} = [H^{1/2}(\Gamma_1)]^d \quad \text{and} \quad Q^*_1 = [H^{1/2}(\Gamma)]^d.
\]

The meaning of the operators \( M_i, D_i, \) and \( K_i \) is as in Section 2.2. Note that the material properties of the two bodies might differ. In order to define the constraints, we need to define several boundary operators. With the operator

\[
B_1 : V_1 \to Q^*_1, \quad (B_1 u, q)_{Q^*_1, Q^*_1} := \int_{\Gamma_1} u \cdot q \, dx,
\]
i.e., the trace operator on $\Gamma_1$, we can write the boundary condition on $\Gamma_1$ as the operator equation $B_1 u_1(t) = G_1(t)$. Therein, $G_1(t)$ contains the corresponding Dirichlet data. The coupling constraint is more complicated because it involves a time-dependent boundary and thus transformations, as described in Section 2. Therefore, we introduce time-dependent bi-Lipschitz transformations $\Phi_{12}$ and $\Phi_{21}$ which transform the interval $I$ onto the boundaries, i.e.,

$$\Phi_{12}(I, t) = \Gamma_{12}(t), \quad \Phi_{21}(I, t) = \Gamma_{21}(t).$$

An illustration of the two mappings is shown in Figure 3. These two transformations allow to define the corresponding boundary operators

$$B_{12}(t) : V_1 \to Q_{12}^*, \quad \langle B_{12}(t) u, q \rangle_{Q_{12}^*, Q_{12}} := \int_{\Gamma_{12}(t)} u \cdot (q \circ \Phi_{12}(t)^{-1}) | \det \Phi_{12}(t)^{-1} | \, dx$$

and

$$B_{21}(t) : V_2 \to Q_{12}^*, \quad \langle B_{21}(t) u, q \rangle_{Q_{12}^*, Q_{12}} := \int_{\Gamma_{21}(t)} u \cdot (q \circ \Phi_{21}(t)^{-1}) | \det \Phi_{21}(t)^{-1} | \, dx.$$ 

The coupling condition in operator form then reads $B_{12} u_1 - B_{21} u_2 = G_2(t)$. Note that in general $G_2(t) \neq 0$, since the displacement rather than the deformation has to coincide.

![Figure 3. Illustration of the transformations $\Phi_{12}(t)$ and $\Phi_{21}(t)$, which map the interval $I$ onto $\Gamma_{12}(t)$ and $\Gamma_{21}(t)$, respectively.](image)

As in Section 2, we include the two constraints by the Lagrangian method. Then, the problem has the form: determine $u_1(t) \in V_1, u_2(t) \in V_2, \lambda_1(t) \in Q_1, \lambda_1(t) \in Q_{12}$ such that

$$M_1 \ddot{u}_1(t) + D_1 \dot{u}_1(t) + K_1 u_1(t) + B_1^* \lambda_1(t) + B_{12}^* (t) \lambda_{12}(t) = F_1(t) \quad \text{in } V_1^*, (2a)$$

$$B_1 u_1(t) = G_1(t) \quad \text{in } Q_1^*, (2b)$$

$$M_2 \ddot{u}_2(t) + D_2 \dot{u}_2(t) + K_2 u_2(t) - B_{21}^* (t) \lambda_{12}(t) = F_2(t) \quad \text{in } V_2^*, (2c)$$

$$B_{12}(t) u_1(t) - B_{21}(t) u_2(t) = G_2(t) \quad \text{in } Q_{12}^* (2d)$$

holds a.e. in $(t_0, T)$ with given consistent initial conditions for $u_1$, $u_2$, and its first time derivatives. To show that the coupled system $(2)$ has the same structure as $(1)$, we perform a reformulation.

### 3.2 Reformulation

To write the coupled system in the form of $(1)$, we define the spaces

$$V := V_1 \otimes V_2 \quad \text{and} \quad Q := Q_1 \otimes Q_{12}$$

with dual spaces $V^* = V_1^* \otimes V_2^*$ and $Q^* = Q_1^* \otimes Q_{12}^*$. Thus, any function $u \in V$ can be written in the form $u = [u_1, u_2]^T$ with $u_1 \in V_1$ and $u_2 \in V_2$. The corresponding norms are defined by $\|u\|_V = \|u_1\|_{V_1} + \|u_2\|_{V_2}$ and analogously for $\|\cdot\|_Q$. With the introduced notation, for functions $u \in V$ we define the operators

$$M u := \begin{bmatrix} M_1 u_1 \\ M_2 u_2 \end{bmatrix}, \quad D u := \begin{bmatrix} D_1 u_1 \\ D_2 u_2 \end{bmatrix}, \quad K u := \begin{bmatrix} K_1 u_1 \\ K_2 u_2 \end{bmatrix}.$$ 

In the same way we also define the functionals $F \in V^*$ and $G \in Q^*$. To combine the constraints, we define the time-dependent operator

$$B(t) : V \to Q^* \quad \text{by} \quad B(t) u := \begin{bmatrix} B_1 u_1 \\ B_{12} (t) u_1 - B_{21} (t) u_2 \end{bmatrix}.$$ 

Note that since $B_1 u_1(t) \in Q_1^*$ and $B_{12}(t) u_1(t) - B_{21}(t) u_2(t) \in Q_2^*$, the operator $B(t)$ is well-defined.
Theorem 1 (Inf-sup condition). If the operators $B_1$ and $B_{21}$ satisfy an inf-sup condition independently of $t$, then so does the operator $B$. This means there exists a positive constant $\beta$ such that
\[
\inf_{q \in Q} \sup_{v \in V} \frac{\langle B(t)v, q \rangle}{\|v\| \|q\|_Q} \geq \beta > 0.
\]

Proof. Consider an arbitrary $q \in Q$. By the definition of the space $Q$, there exist $q_1 \in Q_1$ and $q_{12} \in Q_{12}$ such that $q = [q_1, q_{12}]^T$. Since the operators $B_1$ and $B_{21}$ satisfy the stability condition with constants $\beta_1$ and $\beta_{21}$, there exist $u_1 \in V_1$ and $u_{21} \in V_2$ such that
\[
\frac{\langle B_1 u_1, q_1 \rangle}{\|u_1\| \|q_1\|_{Q_1}} \geq \frac{\beta_1}{\sqrt{2}}, \quad \frac{\langle B_{21} (t) u_{21}, q_{12} \rangle}{\|u_{21}\| \|q_{12}\|_{Q_{12}}} \geq \frac{\beta_{21}}{\sqrt{2}}.
\]
We consider two cases. If $\|q_1\|_{Q_1} \leq \|q_{12}\|_{Q_{12}}$, then we set $u := [0, -u_2]^T$ and obtain
\[
\langle B(t)u, q \rangle = \langle B_1(0)u_1, q_1 \rangle + \langle B_{21}(t)0, q_{12} \rangle = \langle B_{21}(t)u_2, q_{12} \rangle.
\]

Therewith, we obtain the assertion via
\[
\sup_{v \in V} \frac{\langle B(t)v, q \rangle}{\|v\| \|q\|_Q} \geq \frac{\|B(t)v, q\|}{\|v\| \|q\|_Q} = \frac{\langle B_{21}(t)u_2, q_{12} \rangle}{\|u_{21}\| \|q_{12}\|_{Q_{12}}} \geq \frac{\|B_{21}(t)u_2, q_{12}\|_Q}{\sqrt{2} \|v_2\| \|q_{12}\|_{Q_{12}}} \geq \frac{\beta_{21}}{\sqrt{2}}.
\]

In the other case, i.e., $\|q_1\|_{Q_1} > \|q_{12}\|_{Q_{12}}$, we define $u := [u_1, w]^T$ where $v \in V_2$ satisfies $w \circ \Phi_{21} = u_1 \circ \Phi_{12}$ on the boundary (in $Q_{12}$). Note that, up to this point, $w$ is not uniquely defined since only its trace is fixed. This choice yields $\langle B_{21}(t)w, q_{12} \rangle = \langle B_{21}(t)u_1, q_{12} \rangle$ and thus,
\[
\sup_{v \in V} \frac{\langle B(t)v, q \rangle}{\|v\| \|q\|_Q} \geq \frac{\|B(t)v, q\|}{\|v\| \|q\|_Q} = \frac{\|B_{21}(t)u_1, q_{12}\|_Q}{\|u_{21}\| \|q_{12}\|_{Q_{12}}} \geq \frac{\beta_{21}}{\sqrt{2}}.
\]

The inequality denoted by $(\ast)$ follows from the continuity of $B_{12}$ and the inf-sup condition for $B_{21}$. The latter guarantees the existence of a continuous right-inverse and thus, with a suitable choice of $w$, it follows that there exist positive constants $c_1, c_2 \in \mathbb{R}$ with
\[
\|w\|_{V_2} \leq c_1 \|\text{trace}(w \circ \Phi_{21})\|_{Q_{12}} = c_1 \|\text{trace}(u_1 \circ \Phi_{12})\|_{Q_{12}} \leq c_1 c_2 \|u_1\|_{V_1}.
\]

Remark 2. The proof of Theorem 1 does not require an inf-sup condition for $B_{12}$. For this operator it suffices to have continuity to obtain $\|\text{trace}(u_1 \circ \Phi_{12})\|_{Q_{12}} \leq \|u_1\|_{V_1}$.

With the inf-sup stability of $B$, we have the same properties of the operators as in the single body case. This includes the positive definiteness of the operator $M$. In fact, putting everything together, we obtain the same structure as in system (1). It remains to show that this system is equivalent to the coupled system (2). For this, note that the operator equation is equivalent to a weak formulation with test spaces $V$ and $Q_1$, respectively. Applying test functions of the form $[v_1, 0]^T \in V$ and $[0, v_2]^T \in V$ with $v_1 \in V_1$ and $v_2 \in V_2$, we obtain the weak form of the operator equations (2a) and (2c). Analogously, we obtain the remaining equations (2b) and (2d).

As a result, all comments on the discrete structure and possible index reduction techniques from Section 2.3 remain true for the coupled case.

3.3 Example

As an illustration, we discuss an example motivated by the pantograph and catenary system [2]. We consider a rectangular flexible body $\Omega$ in two space dimensions with zero boundary conditions on two opposed sides, namely $\Gamma_0$, and a moving Dirichlet boundary on the lower side, namely $\Gamma_D(t)$. To focus on the time-dependent condition, we include the zero boundary conditions in the ansatz space, i.e., we define $V := [H^1_{0,\Omega}]^2$. The moving boundary $\Gamma_D(t)$ is moving with constant speed $v_0$ and is coupled with a lumped mass model, as shown in Figure 4. Thus, we consider a coupling of a flexible body with a classical MBS.

If we set the interval $I$ to be $\Gamma_D(0)$, then the transformation $\Phi(t): I \to \Gamma_D(t)$ is defined by
\[
\Phi(t, x, y) = (x + tv_0, y).
\]
Its inverse has the same form with velocity \(-v_0\). Hence, with \(Q^* = [H^{1/2}(I)]^2\) the coupling operator \(B(t) : V \to Q^*\) is given by
\[
\langle B(t)u, q \rangle_{Q^*, Q} := \int_{\Gamma_D(t)} u \cdot (q \circ \Phi(t)^{-1}) \, dx = \int_I (u \circ \Phi(t)) \cdot q \, dx.
\]

Let the operators \(M, D, \) and \(K\) again describe the elastic behaviour of the flexible body. Furthermore, the right-hand side \(F\) includes gravitational forces. The coupled system, consisting of one PDE, two ODEs, and one constraint, has the form
\[
M\ddot{u}(t) + D\dot{u}(t) + Ku(t) + B^*(t)\lambda(t) = F(t) \quad \text{in } V^*,
\]
\[
B(t)u(t) = \mathcal{G}(t, w_1) \quad \text{in } Q^*,
\]
\[
m_1\ddot{w}_1(t) + d_1(\dot{w}_1(t) - \dot{w}_2(t)) + c_1(w_1(t) - w_2(t)) = -\langle B(t)1, \lambda(t) \rangle_{Q^*, Q^*},
\]
\[
m_2\ddot{w}_2(t) + d_2(\dot{w}_2(t) - \dot{w}_1(t)) + c_2w_2(t) - c_1(w_1(t) - w_2(t)) = F_{sp}.
\]
The model of the spring damper system is taken from [2]. Therein, \(w_1, w_2\) equal the vertical coordinates as shown in Figure 4, \(m_1, m_2\) denote the masses, \(c_1, c_2\) the spring constants, and \(d_1, d_2\) the damping constants. \(F_{sp}\) equals some force which acts upwards. In the original pantograph and catenary system, this force would correspond to the force of the pantograph in order to retain contact to the wire. The right-hand side \(\mathcal{G}\) is defined by
\[
\langle \mathcal{G}(t, w_1), q \rangle_{Q^*, Q} := w_1 \int_I \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \cdot q \, dx.
\]
This means that the vertical deformation is coupled with the spring-damper system by \(w_1\) and the deformation in horizontal direction should vanish. The coupling term in the ODE is hence given by
\[
\langle B(t)1, \lambda(t) \rangle_{Q^*, Q^*} := \int_I \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \cdot \lambda(t) \, dx.
\]

Therein, \(1\) denotes a function in \(V\) which is constant along \(\Gamma_D(t)\) with value 0 in the first component and 1 in the second.

Some numerical results are given in Figure 5. Within the computations, only vertical deformations were considered. As spatial discretization finite elements with edge-bubble functions are used which satisfy an discrete stability condition [4]. In this scheme, the Lagrange multipliers are modeled as piecewise constants. The Newmark scheme was applied for the temporal discretization.

### 4 Conclusion

We have shown how to couple flexible MBSs with the help of Dirichlet boundary conditions. Using a bi-Lipschitz transformation, we were able to allow for constraints on parts of the boundary which change within time. An analysis of the coupling condition proved that the structure is preserved from the single body case, in which the boundary condition is enforced as a weak constraint. We therefore also showed the inf-sup stability of the coupling condition.

The given example from Section 3.3 indicates that the coupling is not restricted to interconnections of boundaries. It allows to couple arbitrary systems from multiphysics to flexible bodies through Dirichlet conditions.
Figure 5. Numerical results of the example from Section 3.3: The plot on the left shows the vertical deformation at the lower boundary of $\Omega$ after 0.3 seconds. The part of the boundary with the Dirichlet boundary condition is colored red. The plots on the right show the vertical displacements $w_1$, $w_2$ within the spring-damper system over time, cf. Figure 4.

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References


