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# Structural-Algebraic Remodeling of Coupled Dynamical Systems*

Lena Scholz and Andreas Steinbrecher[1]

***Abstract*— The automated modeling of multi-physical dynamical systems is usually realized by the coupling of different subsystems via certain interface or coupling conditions. This approach results in large-scale high-index differential-algebraic equations (DAEs). Since the direct numerical simulation of these kinds of systems leads to instabilities and possibly non-convergence of the numerical methods a regularization or remodeling of the system is required. In many simulation environments a kind of structural analysis based on the sparsity pattern of the system is used to determine the index and a reduced system model. However, this approach is not reliable for certain problem classes, in particular not for coupled systems of DAEs. We will present a new approach for the remodeling of coupled dynamical systems that combines the structural analysis, in particular the Signature Method [9], with classical algebraic regularization techniques. This allows to handle so-called structurally singular systems and also enables a proper treatment of redundancies or inconsistencies in the system.**

## I. INTRODUCTION

Modeling and simulation of multi-physics dynamical systems is an important issue in many industrial applications. In modern simulation packages like DYMOLA, MATLAB/SIMULINK, MAPLESIM, SIMULATIONX etc., the modeling of multi-physics systems is automated using a hierarchical network of coupled subsystems. During the compilation process different subsystems (for different physical domains) are coupled together via certain interface or coupling conditions. This way of modeling leads to *differential-algebraic equations (DAEs)*. If the resulting system of DAEs is of high index, then the direct numerical simulation can lead to instabilities and possibly non-convergence of the numerical methods. Thus, often the algebraic constraints are resolved or replaced by their derivatives (i.e., the system is transformed to state-space form), but then the numerical solution can deviate from the constraints or interface conditions and numerical damping or numerical dissipation can occur. Therefore, a regularization or remodeling of the model equations is required to guarantee stable and robust numerical computations, see [1], [4].

In many modeling and simulation tools, the current state of the art to deal with high index DAEs is to use some kind of structural analysis based on the sparsity pattern of the system. Here, generic structural information is used to identify the constraints and interface conditions, to determine the index of the system and to compute an index-reduced regularized system model. However, this approach is not reliable for certain problem classes, in particular not for coupled systems of DAEs as will be explained in the following.

We present a new approach for the remodeling of coupled dynamical systems that combines the structural analysis with classical algebraic regularization techniques. This allows to handle so-called *structurally singular systems*. Moreover, the new approach enables a proper treatment of redundancies or inconsistencies in the system.

## II. COUPLED SYSTEMS OF DAES

We consider coupled systems of DAEs where the dynamics in each subsystem $\mathcal{S}_i$ is given by a DAE in semi-explicit form, i.e.,

$$\mathcal{S}_i : \quad \begin{cases} \dot{x}^i &= f^i(t, x^i, y^i, u^i), \\ 0 &= g^i(t, x^i, y^i, u^i), \end{cases} \tag{1}$$

for $i = 1, \ldots, N$, with differential state variables $x^i : \mathbb{I} \to \mathbb{R}^{n_x^i}$, algebraic state variables $y^i : \mathbb{I} \to \mathbb{R}^{n_y^i}$ and inputs $u^i : \mathbb{I} \to \mathbb{R}^{n_u^i}$, $\mathbb{I} \subset \mathbb{R}$, of subsystem $\mathcal{S}_i$ respectively. For each subsystem $\mathcal{S}_i$ we assume that for a given input $u^i$ the subsystem is of differentiation index (d-index) $\nu_d = 1$. The d-index is defined as the minimal number of times that all or part of the equations in the system must be differentiated in order to obtain an explicit ordinary differential system for all unknowns (in this case for $x^i$ and $y^i$). For details see also [1]. This means that the Jacobian $g^i_{,y^i} := \frac{\partial g^i}{\partial y^i}$ is nonsingular for all points $(t, x^i, y^i)$ in the set of consistency $\mathbb{M}^i := \{(t, x^i, y^i) \in \mathbb{I} \times \mathbb{R}^{n_x^i} \times \mathbb{R}^{n_y^i} \mid g^i(t, x^i, y^i, u^i) = 0\}$ of the corresponding subsystem $\mathcal{S}_i$ (assuming that $u^i$ is given). The use of these semi-explicit d-index-1 subsystems is justified by the assumption that each subsystem has been regularized in a pre-processing step or modeled directly in a suitable way incorporating the special structure of each uni-physical subcomponent.

In the following, we restrict to cyclic coupling of two subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, i.e., $N = 2$, via coupling conditions given in the form

$$u^i = G_{ij}(x^j, y^j), \quad i, j = 1, 2,\ i \neq j, \tag{2}$$

see Figure 1. Here, the functions $G_{ij} \in C(\mathbb{R}^{n_x^j} \times \mathbb{R}^{n_y^j}, \mathbb{R}^{n_u^i})$ that describe the connection of the states of subsystems $\mathcal{S}_1$ to the input of subsystem $\mathcal{S}_2$ and vice versa are assumed to be sufficiently smooth. Then, the coupled system is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} &= \begin{bmatrix} f^1(t, x^1, y^1, G_{12}(x^2, y^2)) \\ f^2(t, x^2, y^2, G_{21}(x^1, y^1)) \end{bmatrix}, \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} g^1(t, x^1, y^1, G_{12}(x^2, y^2)) \\ g^2(t, x^2, y^2, G_{21}(x^1, y^1)) \end{bmatrix} \end{aligned} \tag{3}$$

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[1]Department of Mathematics, Technische Universität Berlin, Berlin, Germany {lscholz, anst}@math.tu-berlin.de

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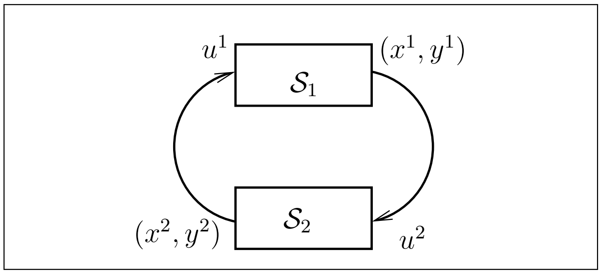


Fig. 1. Cyclic coupling of two subsystems.

or, equivalently, by

$$\begin{aligned} \dot{x} &= f(t,x,y), \\ 0 &= g(t,x,y) \end{aligned} \tag{4}$$

using the notation

$$x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix},\ y = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix},\ f = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix},\ g = \begin{bmatrix} g^1 \\ g^2 \end{bmatrix}, \tag{5}$$

where $x$, $f$ are vector-valued functions of size $n_x = n_x^1 + n_x^2$, and $y$, $g$ are vector-valued functions of size $n_y = n_y^1 + n_y^2$, respectively. Note that a cyclic coupling can easily lead to a high index DAE, even if both subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ are of d-index-1.

*Example 1:* Consider two d-index-1 systems (for given $u^i$)

$$\mathcal{S}_i : \begin{cases} \dot{x}^i &= y^i + b^i(t), \\ 0 &= y^i + u^i + c^i(t), \end{cases} \qquad i = 1,2$$

with $n_x^i = n_y^i = n_u^i = 1$ and smooth forcing functions $b^i(t)$, $c^i(t)$ and coupling conditions

$$u^1 = x^2 + y^2, \quad u^2 = -x^1 + y^1.$$

Then, the coupled system is given by

$$\begin{aligned} \dot{x}^1 &= y^1 + b^1(t), \\ \dot{x}^2 &= y^2 + b^2(t), \\ 0 &= x^2 + y^1 + y^2 + c^1(t), \\ 0 &= -x^1 + y^1 + y^2 + c^2(t). \end{aligned} \tag{6}$$

This DAE is regular (i.e., uniquely solvable) and the d-index is 3. ◁

In general, it holds that the coupled system (4) is regular and of d-index 1 if and only if the Jacobian

$$g_{,y}(t,x,y) = \begin{bmatrix} g^1_{,y^1} & g^1_{,u^1} \cdot G_{12,y^2} \\ g^2_{,u^2} \cdot G_{21,y^1} & g^2_{,y^2} \end{bmatrix} \tag{7}$$

is nonsingular for all points $(t,x,y) \in \mathbb{M} := \{(t,x,y) \in \mathbb{I} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \mid g(t,x,y) = 0\}$. If this condition is not satisfied, then either an increase in the index occurs (that can be arbitrary high), or equations of the form $0 = 0 + \gamma(t)$ are present in the system. In the second case, $\gamma(t) \equiv 0$ leads to redundancies, i.e., algebraic variables are not uniquely determined, or $\gamma(t) \neq 0$ leads to inconsistencies, i.e., the DAE is not solvable anymore. Note that in the last two cases the d-index is not defined anymore and one has to apply more general index concepts as, e.g., the strangeness-index concept, see [4].

## III. REGULARIZATION FOR COUPLED SYSTEMS

Regarding the previous discussion regularization techniques for coupled systems of DAEs are an important issue. Classical regularization techniques for DAEs usually consist of algebraic manipulations, including differentiations, of the system to obtain a reduced system model. These algebraic approaches, e.g., the derivative array approach [2], [4], usually require numerical rank computations and projections onto certain subspaces and thus are computational expensive unless specific structural information can be used. However, algebraic approaches, as the strangeness-index concept [4], allow to deal with over- and underdetermined systems and thus can also be applied to control problems. On the other hand, in many simulation environments often structural approaches are used to reduce the index relying on the sparsity structure of the system. There are various versions and extensions of Pantelides algorithm in combination with the dummy derivative approach [5], [8], or the Signature Method [7], [9]. For these structural approaches fast and efficient algorithms based on graph theoretical concepts exist and in addition the methods provide structural information like a block lower triangular (BLT) form that is useful for further numerical treatments. For regular systems and for many important structures the structural approach usually works well, see [9]. However, it can happen that the approach fails and regular (uniquely solvable) systems cannot be handled, see Section III-A. Moreover, the structural approach usually does not allow to deal with over- and/or underdetermined systems.

In the following, we will present a combined structural-algebraic approach that allows to handle structurally singular systems, i.e., that can be applied when the structural analysis fails, but still uses some information available through the structural analysis. More details can also be found in [11].

### A. Structural Analysis for Coupled Systems

In this section we shortly review the basic steps of the Signature Method ($\Sigma$-method) introduced in [9]. For more details on the $\Sigma$-method see also [6], [7], [9].

The $\Sigma$-method is usually formulated for general nonlinear DAEs of the form

$$F(t, z, \dot{z}) = 0, \tag{8}$$

with sufficiently smooth function $F : \mathbb{I} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. We denote by $F_i$ the components of the vector-valued function $F$ and by $z_j$ the components of the vector-valued function $z : \mathbb{I} \to \mathbb{R}^n$. Then, the $\Sigma$-method consists of the following steps:

1) Built the *signature matrix* $\Sigma = [\sigma_{ij}]_{i,j=1,\ldots,n}$ with

$$\sigma_{ij} := \begin{cases} \text{highest order } k \text{ of derivatives } z_j^{(k)} \text{ in } F_i, \\ -\infty \text{ if } z_j \text{ does not occur in } F_i. \end{cases}$$

2) Find a *highest value transversal (HVT)* of $\Sigma$, i.e., a *transversal* $T$ of $\Sigma$ of the form
$$T = \{(1, j_1), (2, j_2), \dots, (n, j_n)\},$$
where $(j_1, \dots, j_n)$ is a permutation of $(1, \dots, n)$, with maximal value $Val(T) := \sum_{(i,j)\in T} \sigma_{ij}$.
3) Compute the *offset vectors* $c$ and $d$ with $c_i \geq 0$, $d_j \geq 0$ such that
$$\begin{aligned} d_j - c_i &\geq \sigma_{ij} \text{ for all } i, j = 1, \dots, n, \\ d_j - c_i &= \sigma_{ij} \text{ for all } (i, j) \in T. \end{aligned}$$
4) Form the $\Sigma$*-Jacobian* $\mathfrak{J} = [\mathfrak{J}_{ij}]_{i,j=1,\dots,n}$, with
$$\mathfrak{J}_{ij} := \begin{cases} \frac{\partial F_i}{\partial z_j^{(\sigma_{ij})}} & \text{if } d_j - c_i = \sigma_{ij}, \\ 0 & \text{otherwise.} \end{cases}$$
5) Built a *reduced derivative array* $\mathcal{F}(t, \mathcal{Z}) = 0$ consisting of
$$\frac{d^\ell}{dt^\ell} F_i(t, z, \dot{z}) = 0$$
for all $\ell = 0, \dots, c_i$ and for all $i = 1, \dots, n$ with
$$\mathcal{Z} = [z_1, \dot{z}_1, \dots, z_1^{(d_1)}, \dots, \; z_n, \dot{z}_n, \dots, z_n^{(d_n)}]^T.$$
6) *Success check*: if the algebraic system $\mathcal{F}(t^*, \mathcal{Z}^*) = 0$ has a solution $(t^*, \mathcal{Z}^*) \in \mathbb{I} \times \mathbb{R}^{n+\sum_{i=1}^n d_i}$ and $\mathfrak{J}$ is nonsingular at $(t^*, \mathcal{Z}^*)$, then the $\Sigma$-method succeeds.

Note that we call $\mathfrak{J}$ the $\Sigma$-Jacobian since it is in general not the analytical Jacobian, but defined by the offset vectors. Systems for which the $\Sigma$-Jacobian is singular are called *structurally singular* systems. The HVT as well as the offset vectors can be computed by solving a linear assignment problem and the corresponding dual problem, see [9]. For regular systems there always exists an HVT. If no HVT exists, then either some state variables are undetermined or equations of the form $\gamma(t) = 0$ are contained in the system. If the $\Sigma$-method succeeds, it allows to determine the *structural index* of the DAE as
$$\nu_S := \max_i c_i + \begin{cases} 0 & \text{if all } d_j > 0, \\ 1 & \text{if some } d_j = 0. \end{cases}$$
The structural index gives an upper bound for the d-index of the system, i.e., $\nu_d \leq \nu_S$. Equality of the two indices can in general only be shown for specially structured systems, as, e.g., systems in Hessenberg form, see [9].

*Example 2:* If we apply the $\Sigma$-method to the coupled system (6) of Example 1, we get the signature matrix
$$\Sigma = \begin{bmatrix} \boxed{1} & - & 0 & - \\ - & \boxed{1} & - & 0 \\ - & 0 & \boxed{0} & 0 \\ 0 & - & 0 & \boxed{0} \end{bmatrix} \tag{9}$$
with marked HVT on the diagonal and canonical (i.e., smallest possible) offset vectors $c = [0, 0, 0, 0]$ and $d = [1, 1, 0, 0]$. Here, the entries $-$ stand for $-\infty$. The corresponding $\Sigma$-Jacobian is given by
$$\mathfrak{J} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \tag{10}$$
and $\mathfrak{J}$ is singular, i.e., the success check of the $\Sigma$-method fails. If we determine the structural index despite the fact that the success check fails we would get $\nu_S = 1$. However, we have seen that system (6) is a regular system of d-index $\nu_d = 3$. $\triangleleft$

The same observations as in Example 2 can be made for general coupled systems of the form
$$0 = \begin{bmatrix} \dot{x}^1 - f^1(t, x^1, y^1, G_{12}(x^2, y^2)) \\ \dot{x}^2 - f^2(t, x^2, y^2, G_{21}(x^1, y^1)) \\ g^1(t, x^1, y^1, G_{12}(x^2, y^2)) \\ g^2(t, x^2, y^2, G_{21}(x^1, y^1)) \end{bmatrix}. \tag{11}$$
Since the Jacobians $g^i_{,y^i}$ for $i = 1, 2$ are assumed to be nonsingular, we can always reorder the algebraic variables $y^i$ such that the signature matrix is in the form
$$\Sigma = \left[\begin{array}{cccc|cccc} \boxed{1} & \leq 0 & \dots & \leq 0 & & & & \\ \leq 0 & \ddots & \ddots & \vdots & & \leq 0 & & \\ \vdots & \ddots & \ddots & \leq 0 & & & & \\ \leq 0 & \dots & \leq 0 & \boxed{1} & & & & \\ \hline & & & & \boxed{0} & \leq 0 & \dots & \leq 0 \\ & \leq 0 & & & \leq 0 & \ddots & \ddots & \vdots \\ & & & & \vdots & \ddots & \ddots & \leq 0 \\ & & & & \leq 0 & \dots & \leq 0 & \boxed{0} \end{array}\right], \tag{12}$$
with HVT on the main diagonal and canonical offset vectors $c = [0, \dots, 0]$, and $d = [1, \dots, 1, 0, \dots, 0]$. Here, $\leq 0$ denotes entries that are either 0 or $-\infty$. The corresponding $\Sigma$-Jacobian is given by
$$\mathfrak{J} = \left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & \\ 0 & \ddots & \ddots & \vdots & * \\ \vdots & \ddots & \ddots & 0 & \\ 0 & \dots & 0 & 1 & \\ \hline & & 0 & & \frac{\partial g}{\partial y} \end{array}\right], \tag{13}$$
and $\mathfrak{J}$ is nonsingular if and only if $\frac{\partial g}{\partial y}$ is nonsingular. Thus, the $\Sigma$-method succeeds only in the case that the coupled system (4) is again of d-index $\nu_d = 1$!

### B. Combined Structural-Algebraic Approach

In order to be able to treat also structurally singular systems we propose a combined structural-algebraic approach, i.e., we use the $\Sigma$-method in combination with other index reduction techniques to obtain a regularized index-reduced system formulation. Thus, we incorporate the basic ideas of the index reduction by Minimal Extension [3] and the

Dummy Derivative Method [5], that consist of the following steps.

If the $\Sigma$-method succeeds for a given system (8), the canonical offset vector $c$ gives the required information which equations have to be differentiated and how many times in order to be able to extract all hidden constraints. A reduced derivative array can be obtained by adding the derivatives of $F_i$ up to order $c_i$ to the original system. This extended system then consists of $\sum_i c_i + n$ equations in $n$ unknowns $z_1, \ldots, z_n$. More precisely, it depends on $z_1, \dot{z}_1, \ldots, z_1^{(d_1)}, \ldots, z_n, \dot{z}_n, \ldots, z_n^{(d_n)}$, and we have to introduce $\sum_i c_i$ new variables to get the same number of equations and unknowns. For each equation $F_i$, $i = 1, \ldots, n$ we can select a variable $z_{j_i}$ based on the information provided by the HVT, i.e., we choose the unique $j_i$ such that $(i, j_i) \in T$. Then, we introduce the new variables

$$\begin{aligned} &w_{j_i}^{\sigma_{ij_i}+1} \quad \text{replacing} \quad z_{j_i}^{(\sigma_{ij_i}+1)}, \\ &\vdots \\ &w_{j_i}^{\sigma_{ij_i}+c_i} \quad \text{replacing} \quad z_{j_i}^{(\sigma_{ij_i}+c_i)} = z_j^{(d_j)}, \end{aligned} \tag{14}$$

if $c_i > 0$. We collect these new variables in the vectors

$$w_{j_i} := \begin{bmatrix} w_{j_i}^{\sigma_{ij_i}+1} \\ \vdots \\ w_{j_i}^{\sigma_{ij_i}+c_i} \end{bmatrix} \quad \text{for } c_i > 0,$$

and define

$$w_{j_i} := [.] \in \mathbb{R}^0 \quad \text{if } c_i = 0.$$

Note that this replacement of variables is not unique, since it depends on the chosen HVT. In order to choose the best suitable HVT, i.e., one that is valid in a maximal neighborhood of a consistent point, we use a weighting of the HVTs. We define a (local) *weighting coefficient* for each possible HVT by

$$\kappa_T := \prod_{(i,j)\in T} |\mathfrak{J}_{ij}|.$$

If there is more than one possible HVT, we choose one with largest value $\kappa_T$. In the following, we denote by $T^*$ a HVT of $\Sigma$ with $\kappa_{T^*} = \max_T(\kappa_T)$.

*Theorem 3:* Consider a nonlinear DAE (8) with sufficiently smooth function $F : \mathbb{I} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ for which the $\Sigma$-method succeeds with canonical offsets vectors $c = [c_1, \ldots, c_n]$ and $d = [d_1, \ldots, d_n]$. Let $T^*$ be a HVT of the corresponding signature matrix with $\kappa_{T^*} = \max_T(\kappa_T)$. The *augmented system*

$$\mathcal{F}(t, z, \dot{z}, w) = 0, \tag{15}$$

with

$$w = \begin{bmatrix} w_1^T & \ldots & w_n^T \end{bmatrix}^T$$

is obtained by appending the differentiated equations

$$\frac{d^\ell}{dt^\ell} F_i(t, z, \dot{z}) = 0, \quad \ell = 1, ..., c_i, \tag{16}$$

for each equation with $c_i > 0$, $i = 1, ..., n$ to the DAE (8) and introducing new variables

$$w_j = \begin{bmatrix} w_j^{\sigma_{ij}+1} \\ \vdots \\ w_j^{\sigma_{ij}+c_i} \end{bmatrix}$$

for $z_j^{(\sigma_{ij}+1)}, \ldots, z_j^{(\sigma_{ij}+c_i)}$ for the unique $j$ such that $(i, j) \in T^*$ whenever $c_i > 0$, and $w_j = [.]$ for $c_i = 0$. Then the augmented system (15) is (locally) a regular system of d-index $\nu_d = 1$.

*Proof:* The proof is given in [11]. ∎

*Example 4:* To clarify the process we consider the example of the simple pendulum (of mass 1, length 1, and gravity $\mathfrak{g}$) with system equations given by

$$\begin{aligned} F_1(z, \dot{z}) &= \dot{p}_1 - q_1 &&= 0 \\ F_2(z, \dot{z}) &= \dot{p}_2 - q_2 &&= 0 \\ F_3(z, \dot{z}) &= \dot{q}_1 + 2p_1\lambda &&= 0 \\ F_4(z, \dot{z}) &= \dot{q}_2 + 2p_2\lambda + \mathfrak{g} &&= 0 \\ F_5(z, \dot{z}) &= p_1^2 + p_2^2 - 1 &&= 0 \end{aligned} \tag{17}$$

where $z = \begin{bmatrix} p_1 & p_2 & q_1 & q_2 & \lambda \end{bmatrix}^T$. The signature matrix is given by

$$\Sigma = \begin{bmatrix} 1 & - & 0 & - & - \\ - & 1 & - & 0 & - \\ 0 & - & 1 & - & 0 \\ - & 0 & - & 1 & 0 \\ 0 & 0 & - & - & - \end{bmatrix}, \tag{18}$$

where the two possible HVTs are marked by light and dark gray boxes, and the $\Sigma$-method succeeds with canonical offsets $c = [1, 1, 0, 0, 2]$ and $d = [2, 2, 1, 1, 0]$. Thus, the reduced derivative array is given by

$$\begin{bmatrix} \dot{p}_1 - q_1 \\ \ddot{p}_1 - \dot{q}_1 \\ \dot{p}_2 - q_2 \\ \ddot{p}_2 - \dot{q}_2 \\ \dot{q}_1 + 2p_1\lambda \\ \dot{q}_2 + 2p_2\lambda + \mathfrak{g} \\ p_1^2 + p_2^2 - 1 \\ 2p_1\dot{p}_1 + 2p_2\dot{p}_2 \\ 2p_1\ddot{p}_1 + 2\dot{p}_1^2 + 2p_2\ddot{p}_2 + 2\dot{p}_2^2 \end{bmatrix} = 0$$

consisting of 9 equations in 5 unknowns. We choose the HVT $T_1$ marked by the dark gray boxes, assuming that $\kappa_{T_1} = 4p_1^2 \geq \kappa_{T_2} = 4p_2^2$. For the first equation, i.e., for $i = 1$, we have $c_1 = 1$ and $j_1 = 3$ with $(1, j_1) \in T_1$. Therefore, we introduce one new variable $w_3^1$ for the derivative of order $\sigma_{13} + 1 = 1$ of the variable $z_3 \hat{=} q_1$, i.e., $\dot{q}_1$ is replaced by $w_3^1$. For $i = 2$, we have $c_2 = 1$ and $j_2 = 2$ with $(2, j_2) \in T_1$, so we introduce one new variable $w_2^2$ for the derivative of order $\sigma_{22} + 1 = 2$ of the variable $z_2 \hat{=} p_2$, i.e., $\ddot{p}_2$ is replaced by $w_2^2$. For $i = 3$ and $i = 4$, we have $c_3 = c_4 = 0$, so new variables have to be introduced. Finally, for $i = 5$, we have $c_5 = 2$ and $j_5 = 1$ with $(5, j_5) \in T_1$. Therefore, we introduce two new variables $w_1^1$ and $w_1^2$ for the derivative of order $\sigma_{51}+1=1$ and $\sigma_{51}+2=2$ of the variable $z_1\hat{=}p_1$, i.e., $\dot{p}_1$ is replaced by $w_1^1$ and $\ddot{p}_1$ is replaced by $w_1^2$. With these new variables we get the augmented system

$$\begin{bmatrix} w_1^1 - q_1 \\ w_1^2 - w_3^1 \\ \dot{p}_2 - q_2 \\ w_2^2 - \dot{q}_2 \\ w_3^1 + 2p_1\lambda \\ \dot{q}_2 + 2p_2\lambda + \mathfrak{g} \\ p_1^2 + p_2^2 - 1 \\ 2p_1 w_1^1 + 2p_2\dot{p}_2 \\ 2p_1 w_1^2 + 2(w_1^1)^2 + 2p_2 w_2^2 + 2\dot{p}_2^2 \end{bmatrix} = 0 \qquad (19)$$

with unknowns $p_1, p_2, q_1, q_2, \lambda, w_1^1, w_1^2, w_2^2, w_3^1$. Now, this system is regular with d-index $\nu_d = 1$ and the $\Sigma$-method succeeds with nonsingular $\Sigma$-Jacobian (at a consistent point) and structural index $\nu_S = 1$. ◁

From Theorem 3 we can obtain an d-index 1 system (15) that is much better suited for numerical computations. In the following, we call (15) the *regularized system*. Theorem 3 can also be applied if the $\Sigma$-method overestimates the d-index as can be seen in the following example.

*Example 5:* We consider the simple RC-circuit from [10] described by the DAE system

$$\begin{aligned} C\dot{x}_1 - C\dot{x}_2 - x_3 &= 0, \\ -C\dot{x}_1 + C\dot{x}_2 + Gx_2 &= 0, \\ x_1 + V(t) &= 0, \end{aligned}$$

of d-index $\nu_d = 1$. Here, $C$ and $G$ are positive constants describing the capacitance and conductance, $V(t)$ is a smooth function describing the voltage of the voltage source, $x_1$ and $x_2$ are node potentials, and $x_3$ described the current through the capacitor. The $\Sigma$-method succeeds with $c = [0,0,1]$ and $d = [1,1,0]$ and structural index $\nu_S = 2 > \nu_d = 1$. The extended system is obtained by differentiating the last equation once and introducing $w_1^1$ for $\dot{x}_1$ resulting in the augmented system

$$\begin{aligned} -C\dot{x}_2 - x_3 + Cw_1^1 &= 0, \\ C\dot{x}_2 - Gx_2 - Cw_1^1 &= 0, \\ x_1 + V(t) &= 0, \\ w_1^1 + \dot{V}(t) &= 0. \end{aligned}$$

This system is still regular and of d-index $\nu_d = 1$, but now the $\Sigma$-method succeeds with structural index $\nu_S = 1$. ◁

The presented approach of reducing the index of a DAE based on the structural analysis is similar to the *Method of Dummy Derivatives* [5], where *Pantelides Algorithm* [8] is used to determine the number of times each equation has to be differentiated. It has been shown in [9] that Pantelides Algorithm and the Signature Method described above are essentially equivalent in the sense that, if they can both be applied and they both succeed (or converge) they result in the same structural index and the offsets $c_i$ correspond to the number of differentiations for each equation $F_i$ as determined by Pantelides Algorithm. However, while in the Dummy Derivative Method an algorithm (described in [5]) that is based on selecting certain columns of the Jacobian to obtain a regular submatrix is described, and the selection of the variables which derivatives are replaced are based on this algorithm, in our method the selection of variables is directly prescribed by the HVT $T^*$ and the offset vectors. This selection of variables is much easier to achieve and requires no further numerical computations. As a result, the selected variables are not necessarily the same and the two approaches might result in different regularized d-index-1 systems.

Both approaches, the Dummy Derivative Method and Theorem 3, only work for regular (uniquely solvable) systems. However, the advantage of using the Signature Method as in Theorem 3 is the direct success check (i.e., checking the regularity of the $\Sigma$-Jacobian) that allows the use of the results for further treatment. In comparison, Pantelides Algorithm would not converge in cases where the success check of the Signature Method fails.

With Theorem 3 we can handle systems for which the $\Sigma$-method succeeds. In case of a coupled system (4) this only covers d-index-1 system. In the following, we will present a new approach that can be applied in cases where the success check of the $\Sigma$-method fails, i.e., if the system (4) is structurally singular. We assume that the $\Sigma$-method applied to (4) with appropriate reordering of equations and variables yields a signature matrix of the form

$$\left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right] := \left[\begin{array}{c|c} \tilde{\mathcal{I}}_{n_x} & \leq 0 \\ \hline \leq 0 & \tilde{0}_{n_y} \end{array}\right] \qquad (20)$$

where

$$\tilde{\mathcal{I}}_n := \begin{bmatrix} 1 & & \leq 0 \\ & \ddots & \\ \leq 0 & & 1 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

$$\tilde{0}_n := \begin{bmatrix} 0 & & \leq 0 \\ & \ddots & \\ \leq 0 & & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Thus, we have an HVT on the main diagonal with offsets $c = 0$ and $d = [1,\ldots,1,0,\ldots,0]$, and the corresponding $\Sigma$-Jacobian is given by

$$\left[\begin{array}{c|c} \mathfrak{J}_{11} & \mathfrak{J}_{12} \\ \hline \mathfrak{J}_{21} & \mathfrak{J}_{22} \end{array}\right] := \left[\begin{array}{c|c} I_{n_x} & * \\ \hline 0 & g_{,y} \end{array}\right]. \qquad (21)$$

The Jacobian $g_{,y} = \mathfrak{J}_{22}$ is assumed to be singular, i.e., the success check fails. Singularity of the Jacobian $g_{,y}$ means that the system (4) is either of d-index larger than 1, or that the system (4) is non-regular (in particular, if the Jacobian $g_{,x}$ vanishes identically).

Our aim now is to use as much information from the structural analysis as possible, in order to classify the coupled system as regular or non-regular and, consequently, to compute a regularized formulation that can be used for numerical computations. Therefore, at first we have to identify the equations (and differential variables) that are responsible for a d-index higher than 1. In a second step, we have to differentiate these equations and add the differentiated equations to the original system. For this enlarged system we can introduce new algebraic variables (for the identified differential variables) and replace certain derivatives. If the resulting system is still of high index, we can repeat this process iteratively.

In order to reduce the numerical effort we first transform the block $\Sigma_{22}$ of the signature matrix (20) into block upper triangular form, i.e., by row/column permutation we compute a block partitioning such that

$$\tilde{\Sigma}_{22} = \begin{bmatrix} \tilde{\Sigma}_{22}^{11} & \tilde{\Sigma}_{22}^{12} & \dots & \tilde{\Sigma}_{22}^{1p} \\ -\infty & \tilde{\Sigma}_{22}^{22} & & \vdots \\ \vdots & \ddots & \ddots & \tilde{\Sigma}_{22}^{p-1,p} \\ -\infty & \dots & -\infty & \tilde{\Sigma}_{22}^{pp} \end{bmatrix},$$

with square diagonal blocks $\tilde{\Sigma}_{22}^{ii} = \tilde{0}_{\tilde{N}_i}$ of size $\tilde{N}_i \times \tilde{N}_i$ with $\sum_{i=0}^{p} \tilde{N}_i = n_y$. The blocks $\tilde{\Sigma}_{22}^{ij}$ above the block diagonal only have entries $\leq 0$. Note, that we still have an HVT on the diagonal after this permutation. The block $\mathfrak{J}_{22}$ of the $\Sigma$-Jacobian (21) is permuted accordingly to the form

$$\tilde{\mathfrak{J}}_{22} = \begin{bmatrix} \tilde{\mathfrak{J}}_{22}^{11} & \tilde{\mathfrak{J}}_{22}^{12} & \cdots & \tilde{\mathfrak{J}}_{22}^{1p} \\ 0 & \tilde{\mathfrak{J}}_{22}^{22} & & \vdots \\ \vdots & \ddots & \ddots & \tilde{\mathfrak{J}}_{22}^{p-1,p} \\ 0 & \dots & 0 & \tilde{\mathfrak{J}}_{22}^{pp} \end{bmatrix}.$$

In the following, the coupled system permuted and partitioned according to the block form of $\Sigma$ and $\mathfrak{J}$ is denoted by

$$\begin{aligned} \dot{x} &= f(t, x, \tilde{y}_1, \dots, \tilde{y}_p), \\ 0 &= \tilde{g}_1(t, x, \tilde{y}_1, \dots, \tilde{y}_p), \\ &\vdots \\ 0 &= \tilde{g}_p(t, x, \tilde{y}_1, \dots, \tilde{y}_p), \end{aligned}$$

and $\tilde{y}^T = \begin{bmatrix} \tilde{y}_1^T & \dots & \tilde{y}_p^T \end{bmatrix}$, $\tilde{g}^T = \begin{bmatrix} \tilde{g}_1^T & \dots & \tilde{g}_p^T \end{bmatrix}$. Note, that we have only permuted the algebraic variables and algebraic equations. Now, the singularity of $\mathfrak{J}$ corresponds to the singularity of some of the diagonal blocks $\tilde{\mathfrak{J}}_{22}^{ii}$ of $\tilde{\mathfrak{J}}_{22}$ and we denote by

$$\mathbb{J} := \{ i \in \{1, \dots, p\} \mid \det(\tilde{\mathfrak{J}}_{22}^{ii}) = 0 \}$$

the index set of singular blocks $\tilde{\mathfrak{J}}_{22}^{ii}$. Using this information we can proceed as follows:

*Procedure 1:* Given a coupled system (4) with signature matrix $\Sigma =: \Sigma^0$, canonical offset vectors $c$ and $d$, and $\Sigma$-Jacobian $\mathfrak{J} =: \mathfrak{J}^0$, $q = 0$.

1) **Transformation to block form:** We transform the blocks $\Sigma_{22}^q$ and $\mathfrak{J}_{22}^q$ to the block forms $\tilde{\Sigma}_{22}^q$ and $\tilde{\mathfrak{J}}_{22}^q$ and determine the index set $\mathbb{J}^q$ of singular blocks in $\tilde{\mathfrak{J}}_{22}^q$. If $\mathbb{J}^q = \emptyset$ the $\Sigma$-method succeeds with $\nu_S = 1$ and we proceed with Step 7.
2) **Selection of equations:** We need to identify the equations that are responsible for non-regularity or high index. For each block $(\tilde{\mathfrak{J}}_{22}^q)^{jj}$ with $j \in \mathbb{J}^q$ we compute a smooth matrix function $U_j^q(t, x, \tilde{y})$ of pointwise full rank such that the columns of $U_j^q(t, x, \tilde{y})$ form a basis of the null space of $((\tilde{\mathfrak{J}}_{22}^q)^{jj})^T$. Let $U_j^q(t, x, \tilde{y})$ be of size $\tilde{N}_j \times k_j$, i.e., $k_j = \dim(\mathrm{kernel}(((\tilde{\mathfrak{J}}_{22}^q)jj)^T))$.
3) **Construction of the enlarged system:** For each $j \in \mathbb{J}^q$ we have to differentiate the equations determined by
$$U_j^q(t, x, \tilde{y})^T \tilde{g}_j(t, x, \tilde{y}) = 0.$$
Defining
$$h_j(t, x, \tilde{y}, \dot{x}) := \frac{d}{dt} \left( U_j^q(t, x, \tilde{y})^T \tilde{g}_j(t, x, \tilde{y}) \right)$$
for all $j \in \mathbb{J}^q$ we can built up the enlarged system
$$\begin{bmatrix} \dot{x} - f(t, x, \tilde{y}) \\ \tilde{g}(t, x, \tilde{y}) \\ h(t, x, \tilde{y}, \dot{x}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{22}$$
where
$$h(t, x, \tilde{y}, \dot{x}) := [h_j(t, x, \tilde{y}, \dot{x})]_{j \in \mathbb{J}^q}$$
collects all added differentiated equations.
4) **Introduction of new variables:** For each $j \in \mathbb{J}^q$ we have to introduce $k_j$ new variables and replace the derivatives of selected differential variables. The differential variables are selected based on the sparsity pattern of the signature matrix
$$\tilde{\Sigma}_E^q = \begin{bmatrix} \Sigma_{11}^q & \tilde{\Sigma}_{12}^q \\ \tilde{\Sigma}_{21}^q & \tilde{\Sigma}_{22}^q \\ \tilde{\Sigma}_{31}^q & \tilde{\Sigma}_{32}^q \end{bmatrix}$$
of the enlarged system (22). For the block $\tilde{\Sigma}_{31}^q$ of size $K_q \times n_x$ with $K_q = \sum_{j \in \mathbb{J}^q} k_j$, we determine a transversal
$$T_h = \{(1, \ell_1), (2, \ell_2), \dots, (K_q, \ell_{K_q})\},$$
where $\{\ell_1, \dots, \ell_{K_q}\}$ is subset of $\{1, \dots, n_x\}$ with $\ell_i \neq \ell_j$ for $i \neq j \in \{1, \dots, K_q\}$ and the value of $T_h$ is
$$Val(T_h) = \sum_{(i,j) \in T_h} \tilde{\sigma}_{ij} = K_q \quad \text{with } \tilde{\Sigma}_{31}^q = [\tilde{\sigma}_{ij}]_{i,j}.$$
In addition, the condition that the Jacobian $h_{,[\dot{x}_{\ell_1}, \dots, \dot{x}_{\ell_{K_q}}]}$ is nonsingular at the considered consistent point has to be satisfied.
If we cannot find a transversal $T_h$ with $Val(T_h) = K_q$ and nonsingular $h_{,[\dot{x}_{\ell_1}, \dots, \dot{x}_{\ell_{K_q}}]}$ the system is non-regular and we stop the procedure. Otherwise, each occurrence of $\dot{x}_{\ell_m}$ in (22) is replaced by a new variable $w_{\ell_m}$ for $m = 1, \dots, K_q$ leading to the augmented system
$$F^q(t, \dot{x}, x, \tilde{y}, w) = 0, \quad w = \begin{bmatrix} w_{\ell_1} & \dots & w_{\ell_m} \end{bmatrix}^T. \tag{23}$$

5) **Transformation to semi-explicit form:** We transform the augmented system (23) again to semi-explicit form by permuting the remaining differential variables to the front positions and the remaining differential equations to the top, while replacing the (possibly occurring) differential variables in the newly added equations. Thus, we get a semi-explicit augmented system of the form

$$\begin{aligned} \dot{\hat{x}} &= \hat{f}(t, \hat{x}, \hat{y}), \\ 0 &= \hat{g}(t, \hat{x}, \hat{y}), \end{aligned} \tag{24}$$

with reduced number of differential variables $\hat{x}$ and increased number of algebraic variables $\hat{y}$.

6) **Application of $\Sigma$-method to the augmented system:** We apply the $\Sigma$-method to the augmented semi-explicit system (24) and obtain the signature matrix $\Sigma^{q+1}$ and $\Sigma$-Jacobian $\mathfrak{J}^{q+1}$. If the $\Sigma$-method succeeds with $\nu_S \geq 1$ we proceed with Step 7. Otherwise, i.e., if the $\Sigma$-method fails, we go to Step 1 and proceed iteratively with $q$ increased by 1.

7) **Index reduction:** If the $\Sigma$-method succeeds with $\nu_S > 1$ we can use the index reduction proposed in Theorem 3. If $\nu_S = 1$ we are done and we have constructed a *regularized system*.

*Remark 6:* If the $\Sigma$-Jacobian has been nonsingular in the beginning, the Steps 2.-6. are omitted since $\mathbb{J}^0 = \emptyset$. In this case, the semi-explicit augmented system (24) is just the original system (4) and the $\Sigma$-method can be applied successfully. In each iterative step of Procedure 1 the newly introduced variables are purely algebraic, i.e., the number of differential equations is reduced after each iteration and, thus, the procedure terminates after finitely many steps.

*Theorem 7:* Consider a coupled system of DAEs (4) that is composed by coupling two semi-explicit systems of d-index $\nu_d = 1$ via the coupling condition (2). If we apply Procedure 1, then the process terminates after finitely many steps, either with a resulting regular system (24) for which the $\Sigma$-method succeeds, or due to observed redundancies in system (4).

*Proof:* The proof is given in [11]. ∎

*Example 8:* We apply Procedure 1 to the system (6). Here just one 2-by-2 block $\tilde{\Sigma}^0_{22}$ is given and

$$\tilde{\mathfrak{J}}^0_{22} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

We get $U^0_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as a basis of $\operatorname{kernel}((\tilde{\mathfrak{J}}^0_{22})^T)$. Thus, we have to differentiate the equation

$$0 = -x^1 - x^2 - c^1 + c^2,$$

yielding

$$0 = -\dot{x}^1 - \dot{x}^2 - \dot{c}^1 + \dot{c}^2.$$

Appending this equation to the original system (6) yields the enlarged system

$$\begin{aligned} \dot{x}^1 &= y^1 + b^1, \\ \dot{x}^2 &= y^2 + b^2, \\ 0 &= x^2 + y^1 + y^2 + c^1, \\ 0 &= -x^1 + y^1 + y^2 + c^2, \\ \dot{x}^1 + \dot{x}^2 &= \dot{c}^2 - \dot{c}^1. \end{aligned} \tag{25}$$

The signature matrix of this enlarged system is given by

$$\tilde{\Sigma}^0_E = \left[\begin{array}{cc|cc} 1 & - & 0 & - \\ - & 1 & - & 0 \\ \hline - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ \hline 1 & 1 & - & - \end{array}\right] \tag{26}$$

and we can find the transversal $T_h = \{(1,1)\}$ with $h_{,\dot{x}_1} = -1$ nonsingular. Thus, each occurrence of $\dot{x}_1$ is replaced by the new variable $w_1$ and we get

$$\begin{aligned} 0 &= y^1 - w_1 + b^1, \\ \dot{x}^2 &= y^2 + b^2, \\ 0 &= x^2 + y^1 + y^2 + c^1, \\ 0 &= -x^1 + y^1 + y^2 + c^2, \\ \dot{x}^2 &= -w_1 + \dot{c}^2 - \dot{c}^1. \end{aligned} \tag{27}$$

The augmented system (27) is again transformed to semi-explicit form

$$\begin{aligned} \dot{x}^2 &= y^2 + b^2, \\ 0 &= y^1 - w_1 + b^1, \\ 0 &= x^2 + y^1 + y^2 + c^1, \\ 0 &= -x^1 + y^1 + y^2 + c^2, \\ 0 &= y^2 + w_1 - \dot{c}^2 + \dot{c}^1 + b^2. \end{aligned} \tag{28}$$

Now, the $\Sigma$-method applied to this new system yields

$$\Sigma^1 = \left[\begin{array}{c|cccc} \boxed{1} & - & - & 0 & - \\ \hline - & - & \boxed{0} & - & 0 \\ 0 & - & 0 & \boxed{0} & - \\ - & \boxed{0} & 0 & 0 & - \\ - & - & - & 0 & \boxed{0} \end{array}\right]$$

with $c = [0, 0, 0, 0, 0]$, $d = [1, 0, 0, 0, 0]$, and the $\Sigma$-Jacobian is given by

$$\mathfrak{J}^1 = \left[\begin{array}{c|cccc} 1 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right].$$

The block form is given by

$$\tilde{\mathfrak{J}}^1 = \left[\begin{array}{c|c|ccc} 1 & 0 & 0 & -1 & 0 \\ \hline 0 & 1 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right]$$

and we have

$$\tilde{\mathfrak{J}}^1_{22} = \left[\begin{array}{c|ccc} 1 & -1 & -1 & 0 \\ \hline 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

with $\det(\tilde{\mathfrak{J}}^1_{22})^{22} = 0$. A basis of the kernel of $((\tilde{\mathfrak{J}}^1_{22})^{22})^T$ is given by

$$U^1_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

such that we have to differentiate the equation

$$0 = -x^2 + b^1 - c^1 + \dot{c}^1 - \dot{c}^2 + b^2,$$

giving

$$0 = -\dot{x}^2 + \dot{b}^1 - \dot{c}^1 + \ddot{c}^1 - \ddot{c}^2 + \dot{b}^2.$$

The extended system is given by

$$\begin{aligned} \dot{x}^2 &= y^2 + b^2, \\ 0 &= y^1 - w_1 + b^1, \\ 0 &= x^2 + y^1 + y^2 + c^1, \\ 0 &= -x^1 + y^1 + y^2 + c^2, \\ 0 &= y^2 + w_1 - \dot{c}^2 + \dot{c}^1 + b^2, \\ \dot{x}^2 &= \dot{b}^1 - \dot{c}^1 + \ddot{c}^1 - \ddot{c}^2 + \dot{b}^2. \end{aligned}$$

The signature matrix for the extended system is given by

$$\tilde{\Sigma}^1_E = \left[\begin{array}{c|cccc} 1 & - & - & 0 & - \\ \hline - & - & 0 & - & 0 \\ 0 & - & 0 & 0 & - \\ - & 0 & 0 & 0 & - \\ - & - & - & 0 & 0 \\ \hline 1 & - & - & - & - \end{array}\right] \tag{29}$$

and we can find the only transversal $T_h = \{(1,1)\}$ with $h_{,\dot{x}_2} = -1$ such that each occurrence of $\dot{x}_2$ is replaced by the new variable $w_2$ and we get the purely algebraic system

$$\begin{aligned} 0 &= y^2 - w_2 + b^2, \\ 0 &= y^1 - w_1 + b^1, \\ 0 &= x^2 + y^1 + y^2 + c^1, \\ 0 &= -x^1 + y^1 + y^2 + c^2, \\ 0 &= y^2 + w_1 - \dot{c}^2 + \dot{c}^1 + b^2, \\ 0 &= -w_2 + \dot{b}^1 - \dot{c}^1 + \ddot{c}^1 - \ddot{c}^2 + \dot{b}^2. \end{aligned} \tag{30}$$

Now, for this system the $\Sigma$-method succeeds with structural index $\nu_S = 1$. ◁

*Remark 9:* In each iteration of the described procedure the signature matrix for the augmented system can be obtained efficiently from the signature matrix of the previous step. Furthermore, the information on the HVT and on the offsets can be used to speed up the structural analysis for the augmented system. Also the regularity of $\mathfrak{J}$ has to be verified already during the success check of the $\Sigma$-method, thus, the required null space information of $\mathfrak{J}$ is already (partially) available. Thus, the described procedure requires relatively low additional computational effort.

For the coupled system considered in the Examples 1,2 and 8 the behavior of the simulation environments OPENMODELICA, DYMOLA and MAPLESIM has been compared in [11]. Even for this simple problem it shows that these environments can fail to compute a numerical solution due to the structural singularity, while the regularized formulation (30) can be integrated successfully in all environments. For details see [11].

## IV. CONCLUSIONS

Considering simple coupled systems of semi-explicit d-index-1 DAEs (4) has revealed that an index reduction based only on structural information cannot reliably handle these kind of *structurally singular* systems. However, these kinds of algorithms, e.g., Pantelides Algorithm in combination with the Dummy Derivative Method, are used inside many simulation environments as e.g. DYMOLA, OPENMODELICA or MAPLESIM. Our new combined structural-algebraic approach allows to handle structurally singular systems and thus improves the treatment of high-index DAEs. Here, the somewhat expensive computation of projections onto the corresponding subspaces is only required if the structural analysis cannot handle the problem. Using additional information from the structural analysis, even if the success check fails, allows to select equations/variables more efficiently. Moreover, we can detect non-regularities in the system. The application of a similar structural-algebraic approach for more general system, in particular for quasi-linear DAEs, is currently under investigation.

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