Network-based Modeling of Coupled Electro-Mechanical Systems

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Outline

1. Introduction
2. Modeling of coupled electro-mechanical systems
3. Remodeling and index analysis
   - Minimal extension for mechanical subsystems
   - Minimal extension for electrical subsystems
   - The Index of the Coupled System
4. Numerical Example
Modularized network-based modeling of multi-physics systems leads to differential-algebraic equations (DAEs).

In most simulation environments algebraic constraints are resolved to obtain a system in state space form.

- application to large-scale systems not possible (or expensive)
- complicated formulas with doubtful numerical properties
- numerical solution deviates from constraints/interface conditions (e.g. numerical damping or numerical dissipation).

Regularization/remodeling techniques for large-scale high-index DAEs!
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Mechanical subsystems

\[ \begin{align*}
\dot{p} &= v \\
M(p)\dot{v} &= f(t, p, v, u) - G^T(p, u)\lambda \\
0 &= g(p, u)
\end{align*} \] (EoM)

- positions \( p \in \mathbb{R}^{n_p} \), velocities \( v \in \mathbb{R}^{n_p} \), holonomic constraints \( g(p, u) \in \mathbb{R}^{n_c} \), inputs \( u \in \mathbb{R}^{n_u} \), \( f(t, p, v, u) \) applied forces, \( G^T(p, u)\lambda \) constraint forces.
- The mass matrix \( M(p) \) is positive definite and the constraint matrix \( G(p, u) := \frac{\partial g(p, u)}{\partial p} \) has full row rank.

Properties:

- d-index 3 DAE
- hidden constraints on velocity level: \( \frac{d}{dt}g(p, u) = 0 \)
- hidden constraints on acceleration level: \( \frac{d^2}{dt^2}g(p, u) = 0 \)
- consistent initial values for \( p(t_0), v(t_0), \lambda(t_0) \) are required
\[ A_c \frac{d}{dt} q_c(A_c^T \eta, u, t) + A_R r(A_R^T \eta, u, t) + A_L \nu_L + A_{\psi} \nu_{\psi} + A_I I_s(u, t) = 0 \]
\[ \frac{d}{dt} \phi_L(\nu_L, u, t) - A_L^T \eta = 0 \]
\[ A_{\psi}^T \eta - V'_s(u, t) = 0 \]

- for lumped electrical circuit modeled via the Modified Nodal Analysis
- incidence matrix \( A = [A_c \ A_L \ A_R \ A_{\psi} \ A_I] \), node potentials \( \eta \), branch voltages \( \nu^T = [\nu_C^T \ \nu_L^T \ \nu_R^T \ \nu_{\psi}^T \ \nu_I^T] \), branch currents \( \nu^T = [\nu_C^T \ \nu_L^T \ \nu_R^T \ \nu_{\psi}^T \ \nu_I^T] \).
- \( I_s, V'_s \) describe currents of current sources and voltages of voltage sources; \( r \) describes the resistance, \( q_c \) describes the charges of capacitances, \( \phi_L \) describes the fluxes of inductances.
The index of the MNA equations

Theorem (Estévez Schwarz & Tischendorf 2000)

Assume that no loops of voltage sources/cutsets of current sources exists, all circuit elements are passive and controlled sources are not part of CV-loops or LI-cutsets.

1. If the circuit contains no voltage sources and every node is connected with every other node via a capacitive path the DAE has \( d \)-index 0.
2. If there are neither LI-cutsets nor CV-loops the DAE has \( d \)-index 1.
3. If there exist LI-cutsets or CV-loops the DAE has \( d \)-index 2.

\[ \begin{align*}
\Delta & \text{hidden constraints (in case of CV-loops or LI-cutsets)} \\
0 &= Q_{C,R,V}^T (A_L \frac{d}{dt} \nu_L + A_I \frac{d}{dt} I_s(*)) \\
0 &= \bar{Q}_{V-C,\nu}^T (A_{\nu,\text{ind}} \frac{d}{dt} \eta - \frac{d}{dt} V_{\text{ind}}(t))
\end{align*} \]

\[ \begin{align*}
\Delta & \text{The projectors } \bar{Q}_{V-C} \text{ and } Q_{CVR} \text{ can be determined purely based on graph theoretical algorithms.}
\end{align*} \]
Each subsystem $S_i$ with states $x_i \in \mathbb{R}^{n_i}$, inputs $u_i \in \mathbb{R}^{p_i}$ is given by

$$F_i(t, x_i, \dot{x}_i, u_i, \dot{u}_i) = 0$$

coupling of subsystem $S_i$ with $S_{j_1}, \ldots, S_{j_k}$ via the condition

$$u_i = G_{i,j_1,\ldots,j_k}(t, x_{j_1}, \ldots, x_{j_k})$$

$F_i$ and $G_{i,j_1,\ldots,j_k}$ are assumed to be sufficiently smooth.

We restrict to coupled systems consisting only of $S_1$ and $S_2$, and the coupling is done via $u_i = G_{i,j}(t, x_j)$, $i, j = 1, 2$, $i \neq j$. 

![Diagram of coupled systems](image)
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Coupling of subsystems can easily lead to large-scale high-index DAEs

- numerical instabilities can occur
- drift of numerical solution from constraint manifold
- inaccuracies/order reduction of numerical methods, oscillations

Two-step remodeling procedure

1. Each subcomponent is remodeled using index reduction by minimal extension
   - the special structure of the subsystem is incorporated
   - in the resulting index 1 system all explicit and implicit constraints are available (→ initialization easy, no drift of numerical solution)
   - the variables keep their physical meaning

2. The index of the coupled system constructed by coupling the equivalent index-1 formulations is analyzed.
Mechanics: The extended system

\[
\begin{align*}
\dot{p} &= v \\
M(p)\dot{v} &= f(t, p, v, u) - G^T(p, u)\lambda \\
0 &= g(p, u) \\
0 &= g^I(p, v, u, \dot{u}) \\
0 &= g^{II}(p, v, \lambda, u, \dot{u}, \ddot{u})
\end{align*}
\]  
(EoMx)

where

\[
\begin{align*}
g^I(p, v, u, \dot{u}) := \frac{d}{dt} g(p, u), \\
g^{II}(p, v, \lambda, u, \dot{u}, \ddot{u}) := \frac{d^2}{dt^2} g(p, u)
\end{align*}
\]

i.e., an overdetermined system of equations.

We can (locally) determine an orthogonal matrix \( \Pi_p \in \mathbb{R}^{n_p,n_p} \) s.t.

\[
G(p, u)\Pi_p = \begin{bmatrix} G_1 & G_2 \end{bmatrix},
\]

with \( G_2 \in \mathbb{R}^{nc,nc} \) square and nonsingular

and partition \( \Pi_p^T p = \begin{bmatrix} p_1 \\
p_2 \end{bmatrix}, \quad \Pi_p^T v = \begin{bmatrix} v_1 \\
v_2 \end{bmatrix} \).
Introducing new variables $w_1$ for $\dot{p}_2$ and $w_2$ for $\dot{v}_2$ gives

\[
\begin{align*}
\dot{p}_1 &= v_1 \\
w_1 &= v_2
\end{align*}
\]

\[
M(p)\Pi_p \begin{bmatrix} \dot{v}_1 \\ w_2 \end{bmatrix} = f(t, p, v, u) - G^T(p, u)\lambda \\
0 &= g(p, u) \\
0 &= g^I(p, v, u, \dot{u}) \\
0 &= g^{II}(p, v, \lambda, u, \dot{u}, \ddot{u})
\]

(EoM1)

Properties:

▷ d-index 1
▷ no hidden constraints
▷ no drift of the solution manifold
▷ equivalent formulation, i.e. same solution for $(p, v, \lambda)$.
0 = A_C \frac{d}{dt} q_C(A_C^T \eta, t) + A_R g(A_R^T \eta, t) + A_L \nu_L + A_v \nu_v + A_I I_s(*)

0 = \frac{d}{dt} \phi_L(\nu_L, t) - A_L^T \eta

0 = A_v^T \eta - V' s(*)

0 = \bar{Q}_{v-C}^T A_{v, ind} \frac{d}{dt} \eta - \bar{Q}_{v-C}^T \frac{d}{dt} \nu_{ind}(t)

0 = Q_{cR,v}^T A_L \frac{d}{dt} \nu_L + Q_{cR,v}^T A_I \frac{d}{dt} I_s(*)
Electrics: The extended system

\[ 0 = A_C \mathcal{C}(.) A_C^T \frac{d}{dt} \eta + A_C \eta_{C,t} + A_R g(A_R^T \eta, t) + A_L \eta_L + A_V \eta_V + A_I I_s(*) \]

\[ 0 = L(.) \frac{d}{dt} \eta_L + \phi_L, t - A_L^T \eta \]

\[ 0 = A_V^T \eta - \mathcal{V}_s(*) \]

\[ 0 = \tilde{Q}_{\nu-C}^T A_v^{\text{ind}} \frac{d}{dt} \eta - \tilde{Q}_{\nu-C}^T \frac{d}{dt} \mathcal{V}_{\text{ind}}(t) \]

\[ 0 = Q_{\mathcal{C} \mathcal{R} \nu}^T A_L \frac{d}{dt} \eta_L + Q_{\mathcal{C} \mathcal{R} \nu}^T A_I \frac{d}{dt} I_s(*) \]

using \( \mathcal{C}(.) := \frac{\partial q_C}{\partial \nu_C} \), \( q_{C,t}(.) := \frac{\partial q_C}{\partial t} \), \( L(.) := \frac{\partial \phi_L}{\partial \eta_L} \), \( \phi_L, t(.) := \frac{\partial \phi_L}{\partial t} \).

For hidden constraints due to CV-loops:

- Find transformation \( \Pi \eta \) (permutation!) such that

\[ \tilde{Q}_{\nu-C}^T A_v^{\text{ind}} \Pi^{-1} \eta = [F_1 F_2], \quad F_1 \text{ nonsingular.} \]

- Partition and introducing the new variable

\[ \tilde{\eta} = \Pi \eta \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}, \quad \hat{\eta}_1 = \frac{d}{dt} \eta_1 \]
\[ 0 = \tilde{A}_C C(\cdot)\tilde{A}_c^T \begin{bmatrix} \hat{\eta}_1 \\ \frac{d}{dt} \tilde{\eta}_2 \end{bmatrix} + \tilde{A}_c q_{c, t}(\cdot) + \tilde{A}_R g(\cdot) + \tilde{A}_L \tilde{\eta}_L + \tilde{A}_V \tilde{\eta}_V + \tilde{A}_I I_s(\ast) \]

\[ 0 = \mathcal{L}(v_L, t) \frac{d}{dt} v_L + \phi_{L, t}(v_L, t) - \tilde{A}_L^T \tilde{\eta} \]

\[ 0 = \tilde{A}_V \tilde{\eta} - V_s(\ast) \]

\[ 0 = Q_{CR,V}^T A_L \frac{d}{dt} v_L + Q_{CR,V}^T A_I \frac{d}{dt} I_s(\ast) \]

\[ 0 = F_1 \hat{\eta}_1 + F_2 \frac{d}{dt} \eta_2 - Q_{V-C}^T \frac{d}{dt} V_{ind}(t) \]

(with \( \tilde{A}_* := \Pi_{\eta} A_* \))

▷ Order the nodes and branches such that

\[ Q_{CR,V}^T A_L = \begin{bmatrix} \tilde{A}_L & 0 & I \end{bmatrix}, \quad Q_{CR,V}^T A_I = \begin{bmatrix} \tilde{A}_{I, \text{ind}} & 0 \end{bmatrix} \]

▷ Using this splitting we can rewrite the hidden constraints as

\[ 0 = \tilde{A}_L \frac{d}{dt} \tilde{\eta}_L + \frac{d}{dt} \hat{\eta}_L + \tilde{A}_{I, \text{ind}} \frac{d}{dt} I_{\text{ind}}(t). \]
Electrics: Index-1 formulation

$$0 = \tilde{A}_c C(.) \tilde{A}_c^T \begin{bmatrix} \dot{\tilde{\eta}}_1 \\ \frac{d}{dt} \dot{\tilde{\eta}}_2 \end{bmatrix} + \tilde{A}_c q_c, t(.) + \tilde{A}_R g(.) + \tilde{A}_L \dot{\tilde{\imath}}_L + \tilde{A}_V \dot{\tilde{\imath}}_V + \tilde{A}_I I_s(*)$$

$$0 = \mathcal{L}(.) \begin{bmatrix} \frac{d}{dt} \dot{\tilde{\imath}}_L \\ \frac{d}{dt} \dot{\tilde{\imath}}_L & -\tilde{A}_L \frac{d}{dt} \dot{\tilde{\imath}}_L - \tilde{A}_I, \text{ind} \frac{d}{dt} I_{\text{ind}}(t) \end{bmatrix} + \phi_{L, t(.)} - A_L^T \eta$$

$$0 = \tilde{A}_V^T \dot{\tilde{\eta}} - \mathcal{V}_s(*)$$

$$0 = F_2 \frac{d}{dt} \eta_2 + F_1 \dot{\tilde{\eta}}_1 - Q_{\psi, c}^T \frac{d}{dt} \mathcal{V}_{\text{ind}}(t)$$

Properties:

- d-index 1
- no hidden constraints
- no drift from the solution manifold
- equivalent to MNA equations (i.e. same solution for $\eta, \dot{\imath}_L, \dot{\imath}_V$)
Index analysis for the coupled system

- Consider the equivalent index-1 formulation of subsystem $S_i$:

$$
\begin{align*}
\dot{x}_{i,1} &= k_1^i(t, x_{i,1}, x_{i,2}, u_i) \\
0 &= k_2^i(t, x_{i,1}, x_{i,2}, u_i, \dot{u}_i)
\end{align*}
$$

with $\frac{\partial k_2^i}{\partial x_{i,2}}$ nonsingular,

- and the coupled system

$$
\begin{align*}
\dot{X}_1 &= K_1(t, X_1, X_2), \\
0 &= K_2(t, X_1, X_2, \dot{X}_1, \dot{X}_2),
\end{align*}
$$

with $X_1 = \begin{bmatrix} x_{1,1} \\ x_{2,1} \end{bmatrix}$, $X_2 = \begin{bmatrix} x_{1,2} \\ x_{2,2} \end{bmatrix}$ and

$$
\begin{align*}
K_1(.) &= \begin{bmatrix} k_1^1(t, x_{1,1}, x_{1,2}, G_{1,2}(t, x_{2,1}, x_{2,2})) \\
& k_1^2(t, x_{2,1}, x_{2,2}, G_{2,1}(t, x_{1,1}, x_{1,2})) \end{bmatrix}, \\
K_2(.) &= \begin{bmatrix} k_2^1(t, x_{1,1}, x_{1,2}, G_{1,2}(t, x_{2,1}, x_{2,2}), \frac{d}{dt} G_{1,2}(t, x_{2,1}, x_{2,2})) \\
& k_2^2(t, x_{2,1}, x_{2,2}, G_{2,1}(t, x_{1,1}, x_{1,2}), \frac{d}{dt} G_{2,1}(t, x_{1,1}, x_{1,2})) \end{bmatrix}.
\end{align*}
$$
Then, the coupled system (1) is of d-index 0 (i.e. an ODE), iff

\[ K_2; \dot{x}_2 = \begin{bmatrix} k^1_{2;\dot{x}_12} & k^1_{2;\dot{x}_22} \\ k^2_{2;\dot{x}_12} & k^2_{2;\dot{x}_22} \end{bmatrix} \text{ is nonsingular.} \]

If \( [K_2;\dot{x}_1 \ K_2;\dot{x}_2] \equiv 0 \), then the coupled system is of d-index 1, iff

\[ K_2; x_2 = \begin{bmatrix} k^1_{2;x_12} & k^1_{2;x_22} \\ k^2_{2;x_12} & k^2_{2;x_22} \end{bmatrix} \text{ is nonsingular.} \] (2)

If no derivatives of inputs occur (2) is always fulfilled (coupling only via differential variables or no loops over algebraic variables)

In general an increase of the index will occur if the input function is differentiated.
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Numerical Example - The Dynamo

\[ \dot{x} = v \]
\[ \dot{y} = w \]
\[ m\dot{v} = -\sin(\alpha)\lambda - \cos(\alpha)F_w(\varphi, \dot{\varphi}) \]
\[ m\dot{w} = -mg - \cos(\alpha)\lambda + \sin(\alpha)F_w(\varphi, \dot{\varphi}) \]
\[ 0 = \sin(\alpha)x + \cos(\alpha)y \]

\[ C\dot{\eta} = -G\eta - \iota \]
\[ 0 = \eta - V_s(t, \varphi, \dot{\varphi}) \]

The coupled system has d-index 4!
Numerical Example - The Dynamo

\[
\begin{align*}
\dot{x} &= v \\
\dot{y} &= w \\
m\dot{v} &= -\sin(\alpha) \lambda - \cos(\alpha) F_w(\varphi, \iota) \\
m\dot{w} &= -mg - \cos(\alpha) \lambda + \sin(\alpha) F_w(\varphi, \iota) \\
0 &= \sin(\alpha)x + \cos(\alpha)y
\end{align*}
\]

d-index 3

\[
\begin{align*}
C\dot{\eta} &= -G\eta - \iota \\
0 &= \eta - V_s(t, \varphi, \dot{\varphi})
\end{align*}
\]

d-index 2

The coupled system has d-index 3!
Numerical Example - The Dynamo

\[ z_1 = v \]
\[ \dot{y} = w \]
\[ mz_2 = -\sin(\alpha)\lambda - \cos(\alpha)F_w(\varphi, \iota) \]
\[ mw = -mg - \cos(\alpha)\lambda + \sin(\alpha)F_w(\varphi, \iota) \]
\[ 0 = \sin(\alpha)x + \cos(\alpha)y \]
\[ 0 = \sin(\alpha)v + \cos(\alpha)w \]
\[ 0 = \sin(\alpha)z_2 + \cos(\alpha)\dot{w} \]

\[ Cz_3 = -G\eta - \iota \]
\[ 0 = \eta - V_s(t, \varphi, \dot{\varphi}) \]
\[ 0 = z_3 - \frac{d}{dt}V_s(t, \varphi, \dot{\varphi}) \]

The coupled system has d-index 1!
Implementation in Modelica: (OpenModelica 1.6.0)

- object-oriented
- equation-based
- modularized modeling
- uses structural analysis to transform the system to state-space form

- For the coupled d-index 3 system the simulation fails! ("Error solving linear system of equations, system is singular.")

- The coupled d-index 1 formulation works!
If the coupled system is of d-index 1 numerical solution is fine.

If increase of the index due to coupling occurs remodeling is required.

If each subsystem is of d-index 1 a remodeling based on the topological structure of the interconnection network might be possible (open question).

Redundant or inconsistent equations can be created by the coupling resulting in non-regular systems.

The coupling of various subsystems can result in arbitrary high-index DAEs and a throughout analysis is still open.
Thank you for your attention!


