A Structural-Algebraic Approach for Coupled DAE Systems

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1 Motivation

2 Coupled Systems of DAEs
   - The Index of the Coupled Systems

3 Structural-Algebraic Approach for Coupled Systems
   - Structural Analysis for Coupled Systems
   - A new combined approach
Consider subsystem $S_i$ with states $z_i \in \mathbb{R}^{n_i}$, inputs $u_i \in \mathbb{R}^{p_i}$ is given by a system of nonlinear differential-algebraic equations (DAEs)

$$F_i(t, z_i, \dot{z}_i, u_i) = 0$$

coupling of subsystem $S_i$ with subsystems $S_{j_1}, \ldots, S_{j_k}$ via the condition

$$u_i = G_{i,j_1,\ldots,j_k}(t, z_{j_1}, \ldots, z_{j_k})$$

all functions $F_i$ and $G_{i,j_1,\ldots,j_k}$ are assumed to be sufficiently smooth.
Example - The bicycle dynamo

Electro-mechanical coupling:

\[
\begin{align*}
\dot{x} &= v \\
\dot{y} &= w \\
mg\dot{v} &= -\sin(\alpha)\lambda - \cos(\alpha)F_w(\varphi, j) \\
mg\dot{w} &= -mg - \cos(\alpha)\lambda + \sin(\alpha)F_w(\varphi, j) \\
0 &= \sin(\alpha)x + \cos(\alpha)y \\
C\dot{\eta} &= -G\eta - j \\
0 &= \eta - V_s(t, \varphi, \dot{\varphi})
\end{align*}
\]
Example - The bicycle dynamo

Electro-mechanical coupling:

\[
\begin{align*}
\dot{x} &= v \\
\dot{y} &= w \\
m\dot{v} &= -\sin(\alpha)\lambda - \cos(\alpha)F_w(\varphi, j) \\
m\dot{w} &= -mg - \cos(\alpha)\lambda + \sin(\alpha)F_w(\varphi, j) \\
0 &= \sin(\alpha)x + \cos(\alpha)y
\end{align*}
\]

\[
C\dot{\eta} = -G\eta - j \\
0 = \eta - V_s(t, \varphi, \dot{\varphi})
\]

\[
\eta = V_s \implies \frac{d}{dt}\eta = \frac{d}{dt}V_s \\
C\frac{d}{dt}V_s = -G\eta - j \implies j = -C\frac{d}{dt}V_s - G\eta
\]
Electro-mechanical coupling:

\[
\begin{align*}
\dot{x} &= v \\
\dot{y} &= w \\
m\dot{v} &= -\sin(\alpha)\lambda - \cos(\alpha)F_w(\varphi, j) \\
m\dot{w} &= -mg - \cos(\alpha)\lambda + \sin(\alpha)F_w(\varphi, j) \\
0 &= \sin(\alpha)x + \cos(\alpha)y \\
C\dot{\eta} &= -G\eta - j \\
0 &= \eta - V_s(t, \varphi, \dot{\varphi})
\end{align*}
\]

\[
\begin{align*}
\eta &= V_s \quad \Rightarrow \quad \frac{d}{dt}\eta = \frac{d}{dt}V_s \\
C\frac{d}{dt}V_s &= -G\eta - j \quad \Rightarrow \quad \frac{d}{dt}j = -C\frac{d^2}{dt^2}V_s - G\dot{\eta}
\end{align*}
\]
Electro-mechanical coupling:

\[
\begin{align*}
\dot{x} &= v \\
\dot{y} &= w \\
m\dot{v} &= -\sin(\alpha)\lambda - \cos(\alpha)F_w(\varphi, j) \\
m\dot{w} &= -mg - \cos(\alpha)\lambda + \sin(\alpha)F_w(\varphi, j) \\
0 &= \sin(\alpha)x + \cos(\alpha)y \\
C\dot{\eta} &= -G\eta - j \\
0 &= \eta - V_s(t, \varphi, \dot{\varphi})
\end{align*}
\]

\(\varphi(v)\)

The coupled system has d-index 4!
Large-scale high-index differential-algebraic equations (DAEs) arise. The systems can be over-/underdetermined (redundancies/inputs). The direct numerical solution of high-index systems leads to numerical instabilities; drift of numerical solution from constraint manifold; inaccuracies/order reduction of numerical methods, oscillations.

Often algebraic constraints are resolved (state-space form):
- application to large-scale systems not possible (or expensive);
- numerical solution deviates from constraints/interface conditions
- numerical damping or numerical dissipation

We need regularization/remodeling techniques for large-scale high-index DAEs!
We distinguish between two different approaches:

1. **The Algebraic Approach**: algebraic manipulations (including differentiations) of the system
   - e.g. derivative array approach of Campbell, Kunkel & Mehrmann, ...
   - requires numerical rank computations, projection onto certain subspaces $\rightarrow$ expensive 😞
   - allows to deal with over/underdetermined systems $\rightarrow$ control problems 😊

2. **The Structural Approach**: reduce the index of the system relying on the sparsity structure
   - e.g. Signature Method of Pryce or Pantelides algorithm
     - fast and efficient algorithms based on graph theoretical concepts 😊
     - works for regular systems successfully in many cases, **but not for all** 😞

→ **Can we combine both approaches to improve the treatment of DAEs?**
Outline

1 Motivation

2 Coupled Systems of DAEs
   - The Index of the Coupled Systems

3 Structural-Algebraic Approach for Coupled Systems
   - Structural Analysis for Coupled Systems
   - A new combined approach
Coupling of Index-1 Subsystems

Each subsystem has been remodeled as semi-explicit index-1 system (for given $u^i$)

$$S_i : \begin{cases} \dot{x}^i = f^i(t, x^i, y^i, u^i), \\ 0 = g^i(t, x^i, y^i, u^i), \end{cases} \quad \text{with } g^i_{,y^i} := \frac{\partial g^i}{\partial y^i} \text{ nonsingular}$$

We restrict to cyclic coupling of two subsystems

$$u^i = G_{ij}(x^j, y^j), \quad i, j = 1, 2, \ i \neq j$$

Then, the coupled system is given by

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f^1(t, x^1, y^1, G_{12}(x^2, y^2)) \\ f^2(t, x^2, y^2, G_{21}(x^1, y^1)) \\ g^1(t, x^1, y^1, G_{12}(x^2, y^2)) \\ g^2(t, x^2, y^2, G_{21}(x^1, y^1)) \end{bmatrix} \quad \iff \quad \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} f(t, x, y) \\ g(t, x, y) \end{bmatrix}$$
The coupled system is regular and of index 1 if and only if

\[ g, y(t, x, y) = \begin{bmatrix} g^1_{, y^1} & g^1_{, u^1} \cdot G_{12, y^2} \\ g^2_{, u^2} \cdot G_{21, y^1} & g^2_{, y^2} \end{bmatrix} \text{ is nonsingular} \] (*)

for all points \((t, x, y) \in M = \{(t, x, y) \in I \times \mathbb{R}^d \times \mathbb{R}^a | g(t, x, y) = 0\} \).

If the **Index-1-Condition** (*) is **not** satisfied, then:

- **either** an increase in the index occurs
  - arbitrary high increase in the index is possible
  - the system can also be non-regular
- **or** equations of the form \(0 = 0 + \gamma(t)\) occur, that either lead to redundancies or to inconsistencies (the DAE is not uniquely solvable)
Consider two index-1 systems (for given $u'$)

\[ S_1: \begin{cases} \dot{x}_1 = y_1 + b_1(t), \\ 0 = y_1 + u_1 + c_1(t), \end{cases} \quad S_2: \begin{cases} \dot{x}_2 = y_2 + b_2(t), \\ 0 = y_2 + u_2 + c_2(t), \end{cases} \]

with coupling:

\[ u_1 = x_2 + y_2, \quad u_2 = -x_1 + y_1. \]

The coupled system is given by

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 \\
-1 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2
\end{bmatrix}
+ \begin{bmatrix}
b_1(t) \\
b_2(t) \\
c_1(t) \\
c_2(t)
\end{bmatrix}
\]

- $g, y$ singular,
- the index is increased to 3,
- the coupled system is regular (uniquely solvable).
Motivation

Coupled Systems of DAEs
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Structural-Algebraic Approach for Coupled Systems
  - Structural Analysis for Coupled Systems
  - A new combined approach
The Signature Method (Σ-method) of Pryce (1)

\[ F(t, z, \dot{z}) = 0, \quad F : \mathbb{I} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ suff. smooth} \]

1. Built the signature matrix \( \Sigma = [\sigma_{ij}]_{i,j=1,...,n} \)
   \[
   \sigma_{ij} = \begin{cases} 
   \text{highest derivative of } z_j \text{ in } F_i \\
   -\infty & \text{if } z_j \text{ not in } F_i
   \end{cases}
   \]

2. Find a highest value transversal (HVT) of \( \Sigma \), i.e., a transversal \( T \) of \( \Sigma \)
   \[
   T = \{(1,j_1), (2,j_2), \ldots, (n,j_n)\}, \quad (j_1, \ldots, j_n) \text{ permutation of } (1, \ldots, n)
   \]
   with maximal value \( \text{Val}(T) = \sum_{(i,j) \in T} \sigma_{ij} \).

3. Compute the offsets vectors \( c \) and \( d \) with \( c_i \geq 0 \) such that
   \[
   d_j - c_i \geq \sigma_{ij} \text{ for all } i, j = 1, \ldots, n,
   \]
   \[
   d_j - c_i = \sigma_{ij} \text{ for all } (i,j) \in T.
   \]
4. Form the system \( \Sigma \)-Jacobian \( J = [J_{ij}] \),

\[
J_{ij} = \begin{cases} 
\frac{\partial F_i}{\partial z_j} & \text{if } d_j - c_i = \sigma_{ij} \\
0 & \text{otherwise}
\end{cases}
\]

5. Build an enlarged system of equations \( \mathcal{F}(t, \mathcal{Z}) = 0 \) consisting of

\[
F_i(t, z, \dot{z}) = 0, \quad \frac{d}{dt} F_i(t, z, \dot{z}) = 0, \quad \ldots, \quad \frac{d^{(c_i)}}{dt^{(c_i)}} F_i(t, z, \dot{z}) = 0 \quad \forall i
\]

with \( \mathcal{Z} = [z_1, \dot{z}_1, \ldots, z_1^{(d_1)}, \ldots, z_n, \dot{z}_n, \ldots, z_n^{(d_n)}]^T \).

6. Success check: if \( \mathcal{F}(t, \mathcal{Z}) = 0 \) has a solution \( (t^*, \mathcal{Z}^*) \) and \( J \) is nonsingular at \( (t^*, \mathcal{Z}^*) \), then the method succeeds!
Structural Analysis for the Example

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}^1 \\
\dot{x}^2 \\
\dot{y}^1 \\
\dot{y}^2
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 \\
-1 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x^1 \\
x^2 \\
y^1 \\
y^2
\end{bmatrix}
+ 
\begin{bmatrix}
b^1(t) \\
b^2(t) \\
c^1(t) \\
c^2(t)
\end{bmatrix}
\]

▷ Signature matrix: \( \Sigma = \)

\[
\begin{bmatrix}
1^\bullet & - & 0 & - \\
- & 1^\bullet & - & 0 \\
- & 0 & 0^\bullet & 0 \\
0 & - & 0 & 0^\bullet
\end{bmatrix}
\]

▷ Canonical offsets: \( c = [0, 0, 0, 0] \) and \( d = [1, 1, 0, 0] \)

▷ The \( \Sigma \)-Jacobian

\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & -1 & -1 \\
0 & 0 & -1 & -1
\end{bmatrix}
\]

is singular, the method fails!

→ System is structurally singular, but: DAE is regular/uniquely solvable! 😊
The Σ-Method applied to Coupled Systems (1)

\[ 0 = F(t, x^1, x^2, y^1, y^2, \dot{x}^1, \dot{x}^2) = \begin{bmatrix}
\dot{x}^1 & -f^1(t, x^1, y^1, G_{12}(x^2, y^2)) \\
\dot{x}^2 & -f^2(t, x^2, y^2, G_{21}(x^1, y^1)) \\
g^1(t, x^1, y^1, G_{12}(x^2, y^2)) \\
g^2(t, x^2, y^2, G_{21}(x^1, y^1))
\end{bmatrix} \]

▷ Reorder the algebraic variables \( y^i \) such that

\[
\Sigma = \begin{bmatrix}
1 & \leq 0 \\
\vdots & \vdots \\
\leq 0 & 1 \\
\leq 0 & \leq 0 \\
\leq 0 & \vdots \\
\leq 0 & 0 \\
\end{bmatrix} =: \begin{bmatrix}
\tilde{I}_{nx} & \leq 0 \\
\leq 0 & \tilde{0}_{ny}
\end{bmatrix},
\]

▷ with HVT on main diagonal and \( c = [0, \ldots, 0], \ d = [1, \ldots, 1, 0, \ldots, 0] \).
The $\Sigma$-Jacobian is given by

$$J = \begin{bmatrix} 1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\partial g}{\partial y} \end{bmatrix},$$

$J$ is nonsingular if and only if $\frac{\partial g}{\partial y}$ is nonsingular!

$\Sigma$-method succeeds only if the coupled system is of index 1!
How can we handle structurally singular systems?

- they either may be non-regular containing redundancies
- or regular, but of high index

**Idea:** combined structural-algebraic approach:

- use the structural analysis (e.g. Signature method [Pryce 2001])
- in combination with other index reduction techniques (e.g. index reduction by minimal extension [Kunkel & Mehrmann 2004])
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Consider a general nonlinear DAE of higher index

\[ F(t, z, \dot{z}) = 0, \quad F : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \]

The following process is applied iteratively \((F^0(t, z^0, \dot{z}^0) := F(t, z, \dot{z}))\):

1. Identify all equations responsible for index \(> 1 \mapsto\) subset \(\tilde{F}^i\)
   AND identify differential variables responsible for index \(> 1 \mapsto\) subset \(\tilde{z}^i\).

2. Differentiate the equations \(\tilde{F}^i(t, z^i, \dot{z}^i) = 0\) once and add them to the original system.

3. Introduce new algebraic variables \(w^i\) and replace \(\frac{d}{dt} \tilde{z}^i\) by \(w^i\). Goto 1.
If the $\Sigma$-method succeeds, the offset vector $c$ gives the required information which equations have to be differentiated and how many times.

1. The **extended system** is obtained by adding the derivatives of $F_i$ up to order $c_i$ to the original system ($\leftrightarrow \sum_i c_i + n$ equations in $n$ unknowns)

\[
\mathcal{F}(t, Z) = 0
\]

2. Introduce $\sum_i c_i$ new variables to get a regular “square” system:

- for $i = 1, \ldots, n$ select the unique $j$ such that $(i, j) \in T$ (i.e. the HVT maps equation $F_i$ to variable $z_j^{(\sigma_{ij})}$)
- if $c_i > 0$ introduce new variables $w_{i,1}, \ldots, w_{i,c_i}$ for $z_j^{(\sigma_{ij}+1)}, \ldots, z_j^{(\sigma_{ij}+c_i)} = z_j^{(d_j)}$.

**Note:** this is not the smallest possible number of new variables (not the minimal extension)!
Theorem

Assume that the $\Sigma$-method applied to a general nonlinear high-index DAE

$$F(t, z, \dot{z}) = 0$$

succeeds with highest value transversal $T$ and canonical offset vectors $c$ and $d$. Then, the extended system

$$\mathcal{F}(t, \tilde{Z}, \mathcal{W}) = 0, \quad \mathcal{W} = [w_1^T, \ldots, w_n^T]^T$$

with $\tilde{Z} = Z \setminus \{z_{j}^{(\sigma_{ij}+1)}, \ldots, z_{j}^{(\sigma_{ij}+c_i)}\}$ obtained by appending the derivatives

$$\frac{d^\ell}{dt^\ell} F_i(t, z, \dot{z}) =: F_i^\ell(t, z, \dot{z}, \ldots, z^{(\ell+1)}) \quad \text{for} \quad \ell = 1, \ldots, c_i, \ i = 1, \ldots, n$$

to the DAE and introducing new variables $w_i = [w_{i,1}, \ldots, w_{i,c_i}]^T$ for

$$z_{j}^{(\sigma_{ij}+1)}, \ldots, z_{j}^{(\sigma_{ij}+c_i)}, \ (i, j) \in T, \ i = 1, \ldots n \ with \ c_i > 0,$$

is a regular system of index 1.
The proof of the Theorem has been done for

- semi-explicit systems up to index 2
- equation of motion of multibody systems (index 3)
- Hessenberg systems of size $r$ (index $r$)

- It should also hold true for general (regular) nonlinear systems.
- It can also be applied if the $\Sigma$-method overestimates the index.
Combined Approach: Structurally Singular Systems

Semi-explicit coupled system

\[
\begin{align*}
\dot{x} &= f(t, x, y) \\
0 &= g(t, x, y)
\end{align*}
\]

- **Σ-method with appropriate ordering of equations and variables yields**

\[
\begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix}
:=
\begin{bmatrix}
\tilde{I}_{nx} & \leq 0 \\
\leq 0 & \tilde{0}_{ny}
\end{bmatrix}
\]  
(HVT on diagonal)

- **with Σ-Jacobian**

\[
\begin{bmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{bmatrix}
:=
\begin{bmatrix}
I_{nx} & \ast \\
0 & g_{,y}
\end{bmatrix}
\]

- If \( g_{,y} = J_{22} \) is singular, then the success check of the Σ-method fails
  - if \( g_{,x} \equiv 0 \), the DAE is non-regular,
  - if \( g_{,x} \neq 0 \), the index is \( > 1 \) (possibly non-regular).
Combined Approach: Structurally Singular Systems (1)

If the success check fails we proceed as follows:

1. Identify the equations that have to be differentiated:
   - we have to differentiate equations that lie in \( \text{corange}(J) = \ker(J^T) \)
   - e.g. compute an SVD of the \( \Sigma \)-Jacobian (or QR with pivoting)
     \[
     J = [U_1 \quad U_2] \begin{bmatrix} \tilde{S} & 0 \\ 0 & 0 \end{bmatrix} [V_1^T \quad V_2^T]
     \]
     - the columns of \( U_2 \) form a basis of \( \ker(J^T) \), \( k = \dim(\ker(J^T)) \).
     - For semi-explicit system we only have to compute a basis of \( \ker(J_{22}^T) \).

2. Consider the enlarged system

\[
\begin{bmatrix}
F(t, z, \dot{z}) \\
\frac{d}{dt} \left( U_2^T F(t, z, \dot{z}) \right)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
3. Introduce $k$ new variables $w_1, \ldots w_k$:
   - for semi-explicit systems:
     - select $k$ differential variables in $\text{range}(J_{22})$ (can be obtained from $\Sigma_{21}$),
     - if there are less than $k$ such variables, the system is non-regular.
   - How can we (efficiently) select these variables using information from the structural analysis if the system is not in semi-explicit form?

4. Apply $\Sigma$-method to the extended system:
   - $\Sigma_{new}$ can be obtained efficiently from $\Sigma_{old}$
   - If the $\Sigma$-method succeeds with index $> 1$ we get can use the index reduction by minimal extension, if index $= 1$ we are done.
   - If the method fails, go to 1.

Note: the index is reduced by one after each iteration (1.-4.).
Benefits of the combined structural-algebraic approach:

- The new combined approach allows to handle structurally singular systems.
- The (expensive) computation of projections onto the corresponding subspaces is only required if the structural analysis cannot handle the problem.
- Using additional information from the structural analysis allows to select equations/variables more efficiently.
- We can detect non-regularities in the system.
The procedure for the Example (1)

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}^1 \\
\dot{x}^2 \\
\dot{y}^1 \\
\dot{y}^2
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 \\
-1 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x^1 \\
x^2 \\
y^1 \\
y^2
\end{bmatrix}
+ 
\begin{bmatrix}
b^1 \\
b^2 \\
c^1 \\
c^2
\end{bmatrix}
\]

▷ regular system of index 3

▷ the $\Sigma$-method fails due to singular $\Sigma$-Jacobian

\[
J = 
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & -1 & -1 \\
0 & 0 & -1 & -1
\end{bmatrix}
\]

▷ $U_{22} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is basis of $\ker(J_{22}^T)$.

▷ differentiate the equations

\[
0 = -x^1 - x^2 - c^1 + c^2 \quad \Rightarrow \quad 0 = -\dot{x}^1 - \dot{x}^2 - \dot{c}^1 + \dot{c}^2
\]
The procedure for the Example (2)

Introduce the new variables $w_1$ for $\dot{x}^1$ gives

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\dot{x}^1 \\
\dot{x}^2 \\
\dot{y}^1 \\
\dot{y}^2 \\
\dot{w}_1 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 \\
-1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
\dot{x}^1 \\
\dot{x}^2 \\
\dot{y}^1 \\
\dot{y}^2 \\
\dot{w}_1 \\
\end{bmatrix}
+ 
\begin{bmatrix}
\dot{b}^1 \\
\dot{b}^2 \\
\dot{c}^1 \\
\dot{c}^2 \\
-\dot{c}^1 + \dot{c}^2 \\
\end{bmatrix}
\]

$\Sigma$-method for the new system:

\[
\Sigma_2 = \begin{bmatrix}
- & - & 0^\bullet & - & 0 \\
- & 1^\bullet & - & 0 & - \\
- & 0 & 0 & 0^\bullet & - \\
0^\bullet & - & 0 & 0 & - \\
- & 1 & - & - & 0^\bullet \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
J_2 = \begin{bmatrix}
0 & 0 & 1 & 0 & -1 \\
0 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
-1 & 0 & 1 & 1 & 0 \\
0 & -1 & 0 & 0 & -1 \\
\end{bmatrix}
\]

with $c = [0, 0, 0, 0, 0]$, $d = [0, 1, 0, 0, 0]$, and $J_2$ is still singular!
The procedure for the Example (3)

- Determine the null space of $J_2^T$ (not semi-explicit!) yields $U_2^2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

- i.e. we have to differentiate the equation
  
  \[-x^2 + b^1 + b^2 - c^1 - \dot{c}^1 - \ddot{c}^2 = 0 \quad \Rightarrow \quad -\dot{x}^2 + \dot{b}^1 + \dot{b}^2 - \dot{c}^1 - \ddot{c}^1 - \dddot{c}^2 = 0\]

- introducing a new variable $w_2$ for $\dot{x}^2$ gives

\[
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{y}^1 \\ \dot{y}^2 \\ \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ y^1 \\ y^2 \\ w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} b^1 \\ b^2 \\ c^1 \\ c^2 \\ \tilde{c}^1 \\ \tilde{c}^2 \end{bmatrix}
\]

- Now: for this system the $\Sigma$-method succeeds!
What are the Challenges?

- Realization of the proposed procedure as Software Tool.
- Embedding the concept into network-based modeling environments like Dymola/OpenModelica/MapleSim, etc.
- We have to deal with over/underdetermined systems!
- We have to use of a hierarchical modeling concept (no "global flattening")
- We have to deal with singularly perturbed systems/uncertainties.
- Use information about the coupling network to decide about the index of the system DAEs.
P. Kunkel and V. Mehrmann.  
Index reduction for differential-algebraic equations by minimal extension.  

P. Kunkel and V. Mehrmann.  
*Differential-Algebraic Equations — Analysis and Numerical Solution*.  

DAESA - a Matlab tool for structural analysis of DAEs: Software.  
Technical report CAS-12-01-NN, Department of Computing and Software, McMaster University, Hamilton, ON, Canada, July 2012.

J. Pryce.  
A simple structural analysis method for DAEs.  

Thank you very much for your attention.