Structural-Algebraic Remodeling of Coupled Dynamical Systems

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joint work with A. Steinbrecher

ERC Grant
Modeling, Simulation and Control of Multi-Physics Systems

GAMM 84th Annual Meeting in Novi Sad (Serbia), March 19, 2013
Outline

1 Motivation

2 Regularization Techniques for DAEs
   - Structural Approach: The Signature Method

3 Structural-Algebraic Approach for Coupled Systems
   - Structural Analysis for Coupled Systems
   - Structural-Algebraic Approach for Coupled Systems
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Subsystem $S_i$ with states $z^i \in C^1(\mathbb{I}, \mathbb{R}^{n_i})$, inputs $u^i \in C(\mathbb{I}, \mathbb{R}^{p_i})$, $\mathbb{I} \subset \mathbb{R}$ given by a system of nonlinear differential-algebraic equations (DAEs)

$$F^i(t, z^i, \frac{d}{dt}z^i, u^i) = 0$$

Coupling of subsystem $S_i$ with subsystems $S_{j_1}, \ldots, S_{j_k}$ via the condition

$$u^i = G_{i,j_1,\ldots,j_k}(t, z^{j_1}, \ldots, z^{j_k})$$
Use of **Modelica** as modeling language

- **object-oriented equation-based**
- enables **acausal** modeling

Use of the equation $2x - y = 0$

Simulation environments using **Modelica**

- Dymola (Dassault Systèmes)
- MapleSim (Maplesoft)
- SimulationX (ITI GmbH)
- OpenModelica (OSMC)
- ...
In general the resulting coupled system is a **large-scale high-index DAE**.

The systems can be over-/underdetermined (redundancies/inputs).

Currently this cannot be handled by most **Modelica** compilers.

The direct numerical solution of high-index DAE systems leads to

- numerical instabilities;
- drift of numerical solution from constraint manifold;
- inaccuracies/order reduction of numerical methods, oscillations.

We need regularization/remodeling techniques for the numerical solution of large-scale high-index DAEs!

This should be performed automatically by the **Modelica** compiler.
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   - Structural Approach: The Signature Method

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We distinguish between two different approaches:

1. **The Algebraic Approach**: algebraic manipulations (including differentiations) of the system
   - e.g. derivative array approach of Campbell, Kunkel & Mehrmann, ...
   - requires numerical rank computations, projection onto certain subspaces → expensive 😞
   - allows to deal with over/underdetermined systems → control problems 😊

2. **The Structural Approach**: reduce the index of the system relying on the sparsity structure
   - used in most Modelica simulation environments
   - e.g. Signature Method of Pryce or Pantelides Algorithm
     - fast and efficient algorithms based on graph theoretical concepts 😊
     - works for regular systems successfully in many cases, but not for all 😞

→ Can we combine both approaches to improve the treatment of DAEs?
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Structural Approach: The Signature Method (\(\Sigma\)-method) of Pryce (1)

\[ F(t, z, \dot{z}, \ldots, z^{(p)}) = 0, \quad F : \mathbb{I} \subset \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R}^n \]

1. Built the signature matrix \( \Sigma = [\sigma_{ij}] \)

\[ \sigma_{ij} = \begin{cases} 
\text{highest derivative of } z_j \text{ in } F_i \\
-\infty \text{ if } z_j \text{ not in } F_i 
\end{cases} \]

2. Find a highest value transversal (HVT) of \( \Sigma \), i.e., a transversal \( T \)

\[ T = \{(1,j_1), \ldots, (n,j_n)\}, \]

\((j_1, \ldots, j_n)\) permut. of \((1, \ldots, n)\)

with maximal value \( \text{Val}(T) = \sum_{(i,j) \in T} \sigma_{ij} \).

3. Compute the offsets vectors \( c \) and \( d \) with \( c_i \geq 0 \) such that

\[ d_j - c_i \geq \sigma_{ij} \text{ for all } i, j = 1, \ldots, n, \]

\[ d_j - c_i = \sigma_{ij} \text{ for all } (i, j) \in T. \]

- The offsets are not unique, but there exists a unique "smallest" solution (canonical offsets).
- The canonical offsets are independent of the chosen HVT.
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4. Form the system $\Sigma$-Jacobian $\mathcal{J} = [J_{ij}]$,

$$
\begin{cases}
\frac{\partial F_i}{\partial z_j^{(\sigma_{ij})}} & \text{if } d_j - c_i = \sigma_{ij} \\
0 & \text{otherwise}
\end{cases}
$$

5. Built an enlarged system of equations $\mathcal{F}(t, Z) = 0$ consisting of

$$
\frac{d^{(j)}}{dt^{(j)}} F_i(t, z, \dot{z}) = 0 \quad j = 0, \ldots, c_i \quad \text{for all } i = 1, \ldots, n
$$

with $Z = [z_1, \ldots, z_1^{(d_1)}, \ldots, z_n, \ldots, z_n^{(d_n)}]^T$.

6. Success check: if $\mathcal{F}(t, Z) = 0$ has a solution $(t^*, Z^*)$ and $\mathcal{J}$ is nonsingular at $(t^*, Z^*)$, then the $\Sigma$-method succeeds! $(t^*, Z^*)$ is called a consistent point.

- If the $\Sigma$-method succeeds the structural-index of the DAE is defined by

$$
\nu_S := \max_i c_i + \begin{cases}
0 & \text{if all } d_j > 0, \\
1 & \text{if some } d_j = 0.
\end{cases}
$$

- For the d-index $\nu_d$ it holds that $\nu_d \leq \nu_S$ (often $\nu_d = \nu_S$ [Pryce 2001]).
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6. Success check: if $\mathcal{F}(t, \mathcal{Z}) = 0$ has a solution $(t^*, \mathcal{Z}^*)$ and $\mathcal{J}$ is nonsingular at $(t^*, \mathcal{Z}^*)$, then the $\Sigma$-method succeeds! $(t^*, \mathcal{Z}^*)$ is called a consistent point.

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Each subsystem has been remodeled as **semi-explicit index-1 system** (for given $u^i$)

$$S_i : \begin{cases} \dot{x}^i = f^i(t, x^i, y^i, u^i), \\ 0 = g^i(t, x^i, y^i, u^i), \end{cases}$$

with $g^i_{,y^i} := \frac{\partial g^i}{\partial y^i}$ nonsingular for all points $(t, x^i, y^i)$ in

$$\mathbb{L}^i_0 := \{(t, x^i, y^i, \dot{x}^i) | \dot{x}^i = f^i(t, x^i, y^i, u^i), g^i(t, x^i, y^i, u^i) = 0\}$$

- We restrict to **cyclic coupling of two subsystems**

$$u^i = G_{ij}(t, x^j, y^j), \quad i, j = 1, 2, \ i \neq j$$

- Then, the coupled system is given by

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f^1(t, x^1, y^1, G_{12}(t, x^2, y^2)) \\ f^2(t, x^2, y^2, G_{21}(t, x^1, y^1)) \\ g^1(t, x^1, y^1, G_{12}(t, x^2, y^2)) \\ g^2(t, x^2, y^2, G_{21}(t, x^1, y^1)) \end{bmatrix} \quad \iff \quad \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \begin{bmatrix} f(t, x, y) \\ g(t, x, y) \end{bmatrix}$$
The coupled system is regular and of \textit{d-index 1} if and only if

\[
g_{,y}(t, x, y) = \begin{bmatrix} g_{,y1}^1 & g_{,u1}^1 \cdot G_{12,y2}^1 \\ g_{,u2}^2 \cdot G_{21,y1}^1 & g_{,y2}^2 \end{bmatrix}
\]

is nonsingular \hspace{1cm} (*)

for all points \((t, x, y) \in \mathbb{M} = \{(t, x, y) \in \mathbb{I} \times \mathbb{R}^{nx} \times \mathbb{R}^{ny} | g(t, x, y) = 0 \}.

If the \textit{Index-1-Condition} \((*)\) is not satisfied, then:

\begin{itemize}
  \item either an increase in the index occurs
  \item arbitrary high increase in the index is possible
  \item the system can also be non-regular
  \item or equations of the form \(0 = 0 + \gamma(t)\) occur, that either lead to redundancies or to inconsistencies (the DAE is not uniquely solvable)
\end{itemize}
The coupled system
\[
\begin{align*}
\dot{x} &= f(t, x, y) \\
0 &= g(t, x, y)
\end{align*}
\]
\[x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}, \quad y = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}, \quad f = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix}, \quad g = \begin{bmatrix} g^1 \\ g^2 \end{bmatrix}\]

The coupled system is regular and of d-index 1 if and only if

\[g, y(t, x, y) = \begin{bmatrix}
g^1_{, y^1} & g^1_{, u^1} \cdot G_{12, y^2} \\
g^2_{, u^2} \cdot G_{21, y^1} & g^2_{, y^2}
\end{bmatrix}\]

is nonsingular \((*)\)

for all points \((t, x, y) \in \mathbb{M} = \{(t, x, y) \in I \times \mathbb{R}^{nx} \times \mathbb{R}^{ny} | g(t, x, y) = 0\}\).

If the Index-1-Condition \((*)\) is not satisfied, then:

- either an increase in the index occurs
  - arbitrary high increase in the index is possible
  - the system can also be non-regular
- or equations of the form \(0 = 0 + \gamma(t)\) occur, that either lead to redundancies or to inconsistencies (the DAE is not uniquely solvable)
Consider two index-1 systems (for given $u^i$)

\[
S_1 : \begin{cases} 
\dot{x}_1 = y_1 + b_1(t), \\
0 = y_1 + u_1 + c_1(t), 
\end{cases} \quad S_2 : \begin{cases} 
\dot{x}_2 = y_2 + b_2(t), \\
0 = y_2 + u_2 + c_2(t), 
\end{cases}
\]

with coupling:

\[ u^1 = x^2 + y^2, \quad u^2 = -x^1 + y^1. \]

The coupled system is given by

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{y}_1 \\
\dot{y}_2
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 \\
-1 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
y_1 \\
y_2
\end{bmatrix} + \begin{bmatrix}
b_1(t) \\
b_2(t) \\
c_1(t) \\
c_2(t)
\end{bmatrix}
\]

▷ and $g_y$ is singular,

▷ the d-index is increased to 3,

▷ the coupled system is regular (uniquely solvable for consistent IV).
**Comparison of Solvers for the Example**

**Model (A): interconnected subsystems**

![Diagram of interconnected subsystems]

- Model (A)
  - OpenModelica
  - Dymola
  - MapleSim
  - "System is singular"
  - ok

- Model (B): one DAE systems

\[
\begin{align*}
\dot{x}^1 &= y^1 + b^1(t) \\
\dot{x}^2 &= y^2 + b^2(t) \\
0 &= x^2 + y^1 + y^2 + c^1(t) \\
0 &= -x^1 + y^1 + y^2 + c^2(t)
\end{align*}
\]

**Simulation parameter:**

- Simulation from \( t_0 = 0 \) to \( t_f = 1 \)
- Default solver (Dassl or CK45)
- Consistent initial values for \( x^1(0) \) and \( x^2(0) \) fixed
- \( b^1(t) = \frac{1}{100} e^t, \ c^1(t) = \sin(3t), \ b^2(t) = \frac{1}{1000} e^{-2t}, \ c^2(t) = \cos(t) \)
Comparison of Solvers for the Example

Model (A): interconnected subsystems

\[
\begin{align*}
S_1 & \text{\: interconnected subsystems} \\
S_2 & \text{\: interconnected subsystems}
\end{align*}
\]

Model (B): one DAE systems

\[
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\dot{x}_1 &= y_1 + b^1(t) \\
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The $\Sigma$-Method applied to the Example

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\dot{x}^1 \\
\dot{x}^2 \\
\dot{y}^1 \\
\dot{y}^2 \\
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 \\
-1 & 0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x^1 \\
x^2 \\
y^1 \\
y^2 \\
\end{bmatrix} +
\begin{bmatrix}
b_1(t) \\
b_2(t) \\
c_1(t) \\
c_2(t) \\
\end{bmatrix}
\]

▷ Signature matrix:
\[
\Sigma =
\begin{bmatrix}
1 & - & 0 & - \\
- & 1 & - & 0 \\
- & 0 & 0 & 0 \\
0 & - & 0 & 0 \\
1 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\Rightarrow c = [0, 0, 0, 0], \ d = [1, 1, 0, 0]
\]

▷ The $\Sigma$-Jacobian
\[
\tilde{J} =
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & -1 & -1 \\
0 & 0 & -1 & -1 \\
\end{bmatrix}, \quad \tilde{J}_{ij} := \begin{cases} \frac{\partial F_i}{\partial z_j^{(\sigma_{ij})}} & \text{if } d_j - c_i = \sigma_{ij} \\ 0 & \text{otherwise} \end{cases}
\]

is singular and the method fails!

→ The system is structurally singular, but: DAE is regular/uniquely solvable! 😞
The $\Sigma$-Method applied to Coupled Systems (1)

$$0 = F(t, x, y, \dot{x}) = \begin{bmatrix} \dot{x}^1 - f^1(t, x^1, y^1, G_{12}(t, x^2, y^2)) \\ \dot{x}^2 - f^2(t, x^2, y^2, G_{21}(t, x^1, y^1)) \\ g^1(t, x^1, y^1, G_{12}(t, x^2, y^2)) \\ g^2(t, x^2, y^2, G_{21}(t, x^1, y^1)) \end{bmatrix}$$

$\triangleright$ We can permute the vector $y = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}$ such that

$$\Sigma = \begin{bmatrix} 1 & \leq 0 & \leq 0 \\ \vdots & \vdots & \vdots \\ \leq 0 & 1 & \leq 0 \\ \leq 0 & \leq 0 & 0 \end{bmatrix} , \quad \partial g = \begin{bmatrix} 1 & \cdot & 1 & * \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot \cdot \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \end{bmatrix} ,$$

$\triangleright$ Thus, $\partial g$ is nonsingular if and only if $\partial g / \partial y$ is nonsingular!

$\leftrightarrow$ $\Sigma$-method succeeds only if the coupled system is of d-index 1! ☹️
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Semi-explicit coupled system:
\[ \dot{x} = f(t, x, y) \]
\[ 0 = g(t, x, y) \]

Σ-method with appropriate reordering of equations and variables yields
\[
\begin{bmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{bmatrix}
:=
\begin{bmatrix}
\tilde{I}_{n_x} & \leq 0 \\
\leq 0 & \tilde{0}_{n_y}
\end{bmatrix}
\quad \text{(HVT on diagonal)}
\]
\[
\begin{bmatrix}
\tilde{J}_{11} & \tilde{J}_{12} \\
\tilde{J}_{21} & \tilde{J}_{22}
\end{bmatrix}
:=
\begin{bmatrix}
I_{n_x} & \ast \\
0 & g,\tilde{y}
\end{bmatrix}
\]

where
\[
\tilde{I}_{n_x} = \begin{bmatrix}
1 & \leq 0 \\
\cdot & \cdot & \cdot \\
\leq 0 & 1
\end{bmatrix}
\quad \text{(size } n_x \times n_x),
\tilde{0}_{n_y} = \begin{bmatrix}
0 & \leq 0 \\
\cdot & \cdot & \cdot \\
\leq 0 & 0
\end{bmatrix}
\quad \text{(size } n_y \times n_y)
\]

Assume that \( g,\tilde{y} = \tilde{J}_{22} \) is singular (the success check of the Σ-method fails!)
By permutations we can transform Σ and \( \tilde{J} \) into block lower triangular (BLT) form

\[
\tilde{\Sigma} = \begin{bmatrix}
\tilde{\Sigma}_{11} & -\infty \\
\vdots & \ddots \\
\tilde{\Sigma}_{p1} & \cdots & \tilde{\Sigma}_{pp}
\end{bmatrix}, \quad \tilde{\tilde{J}} = \begin{bmatrix}
\tilde{\tilde{J}}_{11} & 0 \\
\vdots & \ddots \\
\tilde{\tilde{J}}_{p1} & \cdots & \tilde{\tilde{J}}_{pp}
\end{bmatrix},
\]

with blocks \( \tilde{\Sigma}_{ii} \) in one of the three possible forms:

- \( \tilde{\tilde{\Sigma}}_{N_i} \) (Type I)
- \( \tilde{\tilde{\Sigma}}_{0 N_i} \) (Type II)
- \( \tilde{\tilde{\Sigma}}_{I N_i} \leq 0 \) (Type III)

and corresponding blocks \( \tilde{\tilde{J}}_{ii} \) of the form

- \( \tilde{\tilde{\tilde{J}}}_{N_i} \) (Type I)
- \( \tilde{\tilde{\tilde{J}}}_{0 N_i} \) (Type II)
- \( \tilde{\tilde{\tilde{J}}}_{I N_i} \) (Type III)

Blocks \( \tilde{\tilde{J}}_{ii} \) of Type III can be further partitioned \( \leftrightarrow \) refined BLT form of \( \tilde{J} \)

\[
\tilde{J} = \begin{bmatrix}
\tilde{J}_{11} & 0 \\
\vdots & \ddots \\
\tilde{J}_{q1} & \cdots & \tilde{J}_{qq}
\end{bmatrix} \quad \leftrightarrow \quad \tilde{\Sigma} = [\tilde{\Sigma}_{ij}]_{i,j=1:q} \quad \text{(not necessarily BLT)}
\]

Singularity of \( \tilde{J} \) implies the singularity of some blocks \( \tilde{\tilde{J}}_{ii} \)!
**Procedure:** Given $\Sigma =: \Sigma^{(0)}$ with offset vectors $c$ and $d$, and singular $\mathcal{J} =: \mathcal{J}^{(0)}$, $i = 0$.

1. **Refined BLT form:**
   - transform $\Sigma^{(i)}$ and $\mathcal{J}^{(i)}$ to refined BLT forms $\bar{\Sigma}^{(i)}$ and $\bar{\mathcal{J}}^{(i)}$
   - determine the index set $\mathcal{J}$ of singular blocks in $\bar{\mathcal{J}}^{(i)}$

   $$\mathcal{J} := \{ j \in \{1, \ldots, q\} \mid \det(\bar{\mathcal{J}}^{(i)}_{jj}) = 0 \}.$$

2. **Identify equations resp. for structural singularity:** for each $j \in \mathcal{J}$
   - compute orthogonal $U_j$ of size $\bar{N}_j \times k_j$ such that the columns of $U_j$ form a basis of $\text{kernel}(\bar{\mathcal{J}}^{(i)}_{jj}^T)$, i.e., $k_j = \dim(\text{kernel}(\bar{\mathcal{J}}^{(i)}_{jj}^T))$.

3. **Select suitable variables:** for each $j \in \mathcal{J}$
   - based on the sparsity pattern
     $$S_{\bar{\Sigma}^{(i)}_j} := \{(k, \ell) \mid \bar{\sigma}_{k\ell} = 0 \text{ for } \bar{\sigma}_{k\ell} \in \bar{\Sigma}^{(i)}_j\}, \quad \bar{\Sigma}^{(i)}_j := \begin{bmatrix} \bar{\Sigma}^{(i)}_{j1} & \cdots & \bar{\Sigma}^{(i)}_{jj-1} \end{bmatrix}$$
   - select $k_j$ differential variables $\bar{x}_{i_1}, \ldots, \bar{x}_{i_{kj}}$ such that $\bar{d}_\ell = 1$ for each $\ell = i_1, \ldots, i_{kj}$ and there exists $m_\ell$ with $(m_\ell, \ell) \in S_{\bar{\Sigma}^{(i)}_j}$ and $m_{\hat{\ell}} \neq m_\ell$ for $\hat{\ell} \neq \ell$.
   - If there are less than $k_j$ such variables, the system is non-regular. **Stop.**
The Procedure for Structurally Singular Systems (1)

Procedure: Given $\Sigma =: \Sigma^{(0)}$ with offset vectors $c$ and $d$, and singular $\hat{\mathcal{J}} =: \hat{\mathcal{J}}^{(0)}$, $i = 0$.

1. Refined BLT form:
   - transform $\Sigma^{(i)}$ and $\hat{\mathcal{J}}^{(i)}$ to refined BLT forms $\tilde{\Sigma}^{(i)}$ and $\tilde{\mathcal{J}}^{(i)}$
   - determine the index set $\mathcal{J}$ of singular blocks in $\tilde{\mathcal{J}}^{(i)}$

   \[ \mathcal{J} := \{ j \in \{1, \ldots, q\} \mid \det(\tilde{\mathcal{J}}_{jj}^{(i)}) = 0 \}. \]

2. Identify equations resp. for structural singularity: for each $j \in \mathcal{J}$
   - compute orthogonal $U_j$ of size $\tilde{N}_j \times k_j$ such that the columns of $U_j$ form a basis of $\ker((\tilde{\mathcal{J}}_{jj}^{(i)})^T)$, i.e., $k_j = \dim(\ker((\tilde{\mathcal{J}}_{jj}^{(i)})^T))$.

3. Select suitable variables: for each $j \in \mathcal{J}$
   - based on the sparsity pattern
   \[
   S_{\tilde{\Sigma}_j^{(i)}} := \{(k, \ell) \mid \tilde{\sigma}_{k\ell} = 0 \text{ for } \tilde{\sigma}_{k\ell} \in \tilde{\Sigma}_j^{(i)}\}, \quad \tilde{\Sigma}_j^{(i)} := \begin{bmatrix} \tilde{\Sigma}_j^{(i)}_{11} & \ldots & \tilde{\Sigma}_j^{(i)}_{j,j-1} \end{bmatrix}
   \]
   - select $k_j$ differential variables $\tilde{x}_{i_1}, \ldots, \tilde{x}_{i_{kj}}$ such that $\tilde{d}_\ell = 1$ for each $\ell = i_1, \ldots, i_{kj}$ and there exists $m_\ell$ with $(m_\ell, \ell) \in S_{\tilde{\Sigma}_j}$ and $m_\ell \neq m_\hat{\ell}$ for $\hat{\ell} \neq \ell$.
   - If there are less than $k_j$ such variables, the system is non-regular. Stop.
**Procedure:** Given $\Sigma =: \Sigma^{(0)}$ with offset vectors $c$ and $d$, and singular $\widehat{\mathcal{J}} =: \widehat{\mathcal{J}}^{(0)}$, $i = 0$.

1. **Refined BLT form:**
   - transform $\Sigma^{(i)}$ and $\widehat{\mathcal{J}}^{(i)}$ to refined BLT forms $\bar{\Sigma}^{(i)}$ and $\bar{\mathcal{J}}^{(i)}$
   - determine the index set $\mathcal{J}$ of singular blocks in $\bar{\mathcal{J}}^{(i)}$
     \[
     \mathcal{J} := \{ j \in \{1, \ldots, q\} \mid \det(\bar{\mathcal{J}}^{(i)}_{jj}) = 0 \}.
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     \]
   - select $k_j$ differential variables $\tilde{x}_{i_1}, \ldots, \tilde{x}_{i_{k_j}}$ such that $\tilde{d}_\ell = 1$ for each $\ell = i_1, \ldots, i_{k_j}$ and there exists $m_\ell$ with $(m_\ell, \ell) \in S_{\bar{\Sigma}^j}$ and $m_\ell \neq m_\ell$ for $\ell \neq \hat{\ell}$.
   - If there are less than $k_j$ such variables, the system is non-regular. **Stop.**
4. **Built the Enlarged System:** for each \( j \in \mathbb{J} \)

   ▶ differentiate the algebraic equations determined by \( U_j^T \) to built up the enlarged system

\[
\begin{bmatrix}
\dot{x} - f(t, x, y) \\
g(t, x, y) \\
\left[ \frac{d}{dt} \left( U_j^T \tilde{g}(t, \tilde{x}, \tilde{y}) \right) \right]_{j \in \mathbb{J}}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

5. **Introduce new variables:** for each \( j \in \mathbb{J} \)

    ▶ introduce \( k_j \) new variables \( w^j_1, \ldots, w^j_{k_j} \)
    ▶ replace the derivatives \( \tilde{x}_\ell^{(d_\ell)} \) of the selected differential variables by \( w^j_\ell \) (see Step 3.).

6. **Transform the augmented system to semi-explicit form**

7. **Apply the \( \Sigma \)-method to the new system:**

   ▶ If the \( \Sigma \)-method succeeds with \( \nu_S > 1 \): Stop
   ▶ else go to 1. and proceed iteratively.
4. **Built the Enlarged System:** for each $j \in \mathbb{J}$
   - differentiate the algebraic equations determined by $U_j^T$ to built up the enlarged system

$$
\begin{bmatrix}
\dot{x} - f(t, x, y) \\
g(t, x, y) \\
\left[ \frac{d}{dt} (U_j^T \tilde{g}(t, \tilde{x}, \tilde{y})) \right]_{j \in \mathbb{J}}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
$$

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   - If the $\Sigma$-method succeeds with $\nu_S > 1$: **Stop**
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4. **Built the Enlarged System:** for each \( j \in J \)

- Differentiate the algebraic equations determined by \( U_j^T \) to build up the enlarged system

\[
\begin{bmatrix}
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g(t, x, y) \\
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     \[
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     g(t, x, y) \\
     \left[ \frac{d}{dt} (U_j^T \tilde{g}(t, \tilde{x}, \tilde{y})) \right]_{j \in \mathbb{J}}
     \end{bmatrix}
     = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
     \]

5. **Introduce new variables:** for each $j \in \mathbb{J}$
   - introduce $k_j$ new variables $w^j_1, \ldots, w^j_{k_j}$
   - replace the derivatives $\tilde{x}^{(d_\ell)}_\ell$ of the selected differential variables by $w^j_\ell$ (see Step 3.).

6. **Transform the augmented system to semi-explicit form**

7. **Apply the Σ-method to the new system:**
   - If the Σ-method succeeds with $\nu_S > 1$: **Stop**
   - else go to 1. and proceed iteratively.
Properties of the Procedure

- The number of differential variables is decreased after each iteration → the procedure terminates after finitely many steps.
- The newly introduced variables are purely algebraic → consistency and formerly hidden constraints are forced.
- After each step of the procedure the index of resulting system is reduced by 1.
- Non-regularities in the system are detected.

Procedure for the Example

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<thead>
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<th>OpenModelica</th>
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</tbody>
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- Structural analysis cannot handle *structurally singular* problems, even if the system is regular and well-posed.  
  - e.g. coupling of semi-explicit d-index-1 systems

- A regularization/index reduction is not possible based on a structural analysis.

**Benefits of a combined structural-algebraic approach:**

- The new combined approach allows to handle structurally singular systems.
- The (expensive) computation of projections onto the corresponding subspaces is only required if the structural analysis cannot handle the problem and only for small subproblems (refined BLT Partitioning).
- We can detect non-regularities in the system.

**Open problems:**

- Can we use a similar approach if the system is not in semi-explicit form?  
- Can this be efficiently integrated into existing solvers?
P. Kunkel and V. Mehrmann.
Index reduction for differential-algebraic equations by minimal extension.

P. Kunkel and V. Mehrmann.

DAESA - a Matlab tool for structural analysis of DAEs: Software.
Technical report CAS-12-01-NN, Department of Computing and Software, McMaster University, Hamilton, ON, Canada, July 2012.

J. Pryce.
A simple structural analysis method for DAEs.

Thank you very much for your attention.