On a model hierarchy for gas flow simulation

Jeroen J. Stolwijk
Technische Universität Berlin

joint work with Volker Mehrmann

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Both the energy transition and the (gas) crisis in Eastern Europe are important topics today in Europe.

These topics force Europe to use and transport natural gas efficiently.

Figure: Goals of the energy transition in Germany [1].

Sources: gas-roads.eu, mobilexag.de, and nordkurier.de.
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Euler equations: continuity equation, impulse equation, and energy equation. In one dimension:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0,
\]

\[
\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) = -\frac{\lambda}{2D} \rho v |v| - g \rho h', \quad \text{(TA1)}
\]

\[
\frac{\partial}{\partial t} \left( \rho \left( \frac{1}{2} v^2 + e \right) \right) + \frac{\partial}{\partial x} \left( \rho v \left( \frac{1}{2} v^2 + e \right) + pv \right) = -\frac{k_w}{D} (T - T_w),
\]

together with the state equation for real gases

\[
p = R \rho T z(p, T).
\]

Here,
\[
\lambda = \text{pipe friction coefficient}, \ D = \text{diameter}, \ h' = \text{slope of the pipe}, \ e = \text{internal energy}, \ k_w = \text{heat conductivity coefficient}, \ T_w = \text{wall temperature}, \ R = \text{gas constant}, \ z(p, T) = \text{compressibility factor}.
\]
The model hierarchy is used to find an appropriate trade-off between accuracy and computational complexity. Moreover, only the algebraic models are (currently) feasible for optimization.
Figure: Realistic gas network with three compressors, together with the choice of models in the pipelines over time for a moderate accuracy [2].

The simplifications in the hierarchy lead to the algebraic model

\[ \rho v = \rho_{in} v_{in}, \]

\[ p(x) = \sqrt{p_{in}^2 - \frac{\lambda c^2}{2D} \rho v |\rho v|(x - x_0)}, \quad \text{(TA-ALG)} \]

\[ T(x) = (T_{in} - T_w) e^{-\frac{k_w}{Dc_v \rho v} (x - x_0)} + T_w, \]

which is commercially utilized for optimization and now further examined.
Backward error: Computational rounding errors and measurement errors are interpreted as errors/perturbations in the input parameters.

For computing the pressure

$$ p(q) = \sqrt{p_{in}^2 - \frac{\lambda c^2}{2D} \rho v |\rho v| (x - x_0)} $$

the following algorithm can be applied:

\[
\begin{align*}
  z_1 &:= p_{in} \cdot p_{in}; \\
  z_2 &:= c \cdot c; \\
  z_3 &:= \lambda \cdot z_2; \\
  z_4 &:= 2 \cdot D; \\
  z_5 &:= \frac{z_3}{z_4}; \\
  z_6 &:= \rho \cdot v; \\
  z_7 &:= x - x_0; \\
  z_8 &:= z_5 \cdot z_6; \\
  z_9 &:= z_8 \cdot |z_6|; \\
  z_{10} &:= z_9 \cdot z_7; \\
  z_{11} &:= z_1 - z_{10}; \\
  p(q) &:= z_{12} = \sqrt{z_{11}}.
\end{align*}
\]

Every step causes a relative error of machine precision $\varepsilon$. 
Thus,

\[ \tilde{p}(q) = \sqrt{p_{in}^2(1 + \varepsilon_1)(1 + \varepsilon_{11}) - \frac{\lambda c^2(1 + \varepsilon_2)(1 + \varepsilon_3)}{2D(1 - \varepsilon_4)}(1 + \varepsilon_5)\rho v(1 + \varepsilon_6)(1 + \varepsilon_8)|\rho v|} \]
\[ \cdot \sqrt{(1 + \varepsilon_6)(1 + \varepsilon_9)(x - x_0)(1 + \varepsilon_7)(1 + \varepsilon_{10})(1 + \varepsilon_{11})(1 + \varepsilon_{12})} \]
\[ = \sqrt{(p_{in}(1 + \varepsilon_{13}))^2 - \frac{\lambda(1 + \varepsilon_{14})c^2}{2D}\rho v|\rho v|(x - x_0)}. \]

Including measurement errors for the input parameters gives

\[ \tilde{p}(q) = \sqrt{(p_{in}(1 + \varepsilon_{p_{in}})(1 + \varepsilon_{13}))^2 - \frac{\lambda(1 + \varepsilon_\lambda)(1 + \varepsilon_{14})c^2(1 + \varepsilon_c)^2}{2D(1 - \varepsilon_D)}\rho v|\rho v|(1 + \varepsilon_\rho)^2(1 + \varepsilon_\nu)^2(x(1 + \varepsilon_x) - x_0(1 + \varepsilon_{x_0}))} \]
\[ = \sqrt{(p_{in}(1 + \varepsilon_{15}))^2 - \frac{\lambda(1 + \varepsilon_{16})c^2}{2D}\rho v|\rho v|(x(1 + \varepsilon_x) - x_0(1 + \varepsilon_{x_0})).} \]

So the backward error for \( \tilde{p}(q) \) is given by

\[ \tilde{p}(p_{in}, \lambda, x, x_0) = p(p_{in}(1 + \varepsilon_{15}), \lambda(1 + \varepsilon_{16}), x(1 + \varepsilon_x), x_0(1 + \varepsilon_{x_0})). \]
Amplifying Factors

Relative error in the pressure due to perturbations in the data:

\[
\frac{p(q) - p(q + \Delta q)}{p(q)} = \frac{\partial p(q)}{\partial p_{in}} \frac{p_{in}}{p(q)} \Delta p_{in} + \frac{\partial p(q)}{\partial \lambda} \frac{\lambda}{p(q)} \Delta \lambda + \frac{\partial p(q)}{\partial x} \frac{x}{p(q)} \Delta x + \frac{\partial p(q)}{\partial x_0} \frac{x_0}{p(q)} \Delta x_0 + O((\Delta q)^2)
\]

\[
= \left( \frac{p_{in}}{p(q)} \right)^2 \varepsilon_{15} - \frac{\lambda c^2 \rho^2 v^2 (x - x_0)}{4Dp(q)^2} \varepsilon_{16} - \frac{\lambda c^2 \rho^2 v^2 x}{4Dp(q)^2} \varepsilon_x - \frac{\lambda c^2 \rho^2 v^2 x_0}{4Dp(q)^2} \varepsilon_{x_0} + \text{h.o.t.}
\]

Amplifying factors for the relative error \((L = x - x_0)\):

![Amplifying factor A1](chart1.png)

![Amplifying factor A2](chart2.png)
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Univariate Reduced Quadrature (URQ) Method:

\[ \mu_{f_{URQ}}(x_0) = W_0 f(x_0) + \sum_{i=1}^{n} W_i \left[ \frac{f(x_i^+)}{h_i^+} - \frac{f(x_i^-)}{h_i^-} \right] \]

and

\[ \sigma^2_{f_{URQ}}(x_0) = \sum_{i=1}^{n} \left\{ W_i^+ \left[ \frac{f(x_i^+)}{h_i^+} \right]^2 + W_i^- \left[ \frac{f(x_i^-)}{h_i^-} \right]^2 + W_i^\pm \frac{[f(x_i^+)-f(x_0)][f(x_i^-)-f(x_0)]}{h_i^+ h_i^-} \right\} . \]
The theoretical result for the pressure $p(x) = \sqrt{p_{in}^2 - \frac{\lambda c^2 L \rho v |\rho v|}{2D}}$ is illustrated using the URQ Method. The relative standard deviation $\frac{\sigma_x}{\mu_x}$ of the input parameters $x$ is 0.5%.

$\Rightarrow$ A length of 70 km is the absolute maximum (40 km is preferable).
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The error for the pressure in the algebraic model grows quickly if the pipeline length is increased.

This has been shown both theoretically and in a simulation.
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The work of Domschke/Kolb/Lang (2011) shows that a model hierarchy can be used to reach an appropriate trade-off between accuracy and computational complexity.

The algebraic model can be used safely for pipelines up to 70 km length.
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Future Work

▷ Perform the error analysis on the more advanced (time-dependent) models in the hierarchy.
▷ Write a user-friendly MATLAB toolbox that quickly determines which model can be used where in the network, based on an error analysis.
▷ Use model adaptivity to determine which model to use for a desired accuracy.
Any Questions?