Dual pairs of Lyapunov inequalities

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In his inspiring paper [1], *Positive Operators and an Inertia Theorem*, Hans Schneider pointed out a close relationship between inertia theorems for Lyapunov equations and positive operators on the space of Hermitian matrices as well as the theory of M-matrices. Among other things, he showed that Lyapunov's matrix theorem can be extended to the case where a positive operator is added to the Lyapunov operator. This result turned out to be fundamental e.g. for the analysis of linear stochastic systems, see [2]. In typical applications it is interpreted as a criterion for a system to be asymptotically stable.

There is another famous result involving the Lyapunov operator (see e.g. [3, 4]), which plays an important role in model order reduction. For a system $\dot{x} = Ax$ with associated Lyapunov operator $L_A : X \mapsto AX + XA^*$, it can be stated in the following equivalent forms, where

$$\Sigma = \text{diag}\,(\Sigma_1, \Sigma_2) > 0 \text{ with } \sigma(\Sigma_1) \cap \sigma(\Sigma_2) = \emptyset$$

is some block-diagonal matrix.

(a) If $L_A(\Sigma) \leq 0$ and $L_A^*(\Sigma) \leq 0$, then the projected subsystems corresponding to the blocks Σ_i are asymptotically stable.

(b) If $L_A(\Sigma) \leq 0$ and $L_A(\Sigma^{-1}) \leq 0$, then the projected subsystems corresponding to the blocks Σ_i are asymptotically stable.

It is immediate to formulate analogous generalized statements for the case, where a positive operator Π is added to the Lyapunov operator, that is for operators $L_A + \Pi$ as considered in [1]. However, the generalizations of (a) and (b) are no longer equivalent and the proofs are less immediate than the statements. Some of the results appeared recently in [5].

In this talk we discuss applications to model order reduction and show the relation of our results to the theory of positive and cross-positive mappings (e.g. [6, 7, 8]). There are multiple connections to topics treated by Volker in his work.

References

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