Differential-Algebraic Equations, Kronecker form, and Linear Relations

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If we consider the equation

$$E\dot{x}(t) = Ax(t), \tag{1}$$

where E is an invertible $n \times n$ matrix, then the solutions are given via the Jordan canonical form of $E^{-1}A$. If E is not invertible, then (1) may contain purely algebraic equations and it is called a *differential-algebraic equation* (DAE). The investigation of DAEs is one of Volker's favorite research topics.

A characterization of the solutions of (1) requires a generalization of the Jordan canonical form which is the *Kronecker canonical form*, see e.g. [1].

In the case of non-invertible E we can give a meaning to the expression $E^{-1}A$ using linear relations (or, what is the same, subspaces in the product space $\mathbb{C}^n \times \mathbb{C}^n$). Matrices are always identified with linear relations via their graphs. For the general study of linear relations see [2, 3]. Nowadays there is a well developed spectral theory for linear relations including eigenvalues, eigenvectors, Jordan chains, singular chains etc.

We obtain the following surprising observation: The spectral notions (i.e. eigenvalues, eigenvectors, Jordan chains, singular chains, etc.) of the linear relation $E^{-1}A$ correspond in a very natural way to the four block entries of the Kronecker form. This provides a new geometric interpretation of the entries of the Kronecker canonical form.

From another perspective, the Kronecker canonical form can be viewed as the canonical representation for linear relations, similar to the Jordan canonical form being a canonical representation for matrices. Such a canonical representation for linear relations was not known so far.

References

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