Westfälische
Wilhelms-Universität
Münster

Convex Optimization in Imaging

CoSIP Winter Retreat 2016

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WWU Münster

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07.12.2016

Examples from Image Processing

- Image Reconstruction
- Optical Flow

Optimization

- Operator Splitting
- Forward-Backward-Splitting
- Primal-Dual-Method

FlexBox

Examples from Image Processing:

General Consideration

- Finite dimensional optimization problem in $\mathbb{R}^d$

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- Concatenating columns yields vector $u \in \mathcal{X} = \mathbb{R}^{N_1 N_2}$

Variational Model

Introduced problems can be expressed as general Variational Problem

$$\arg\min_{u} D(u, f) + \alpha R(u)$$

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Regularization:

- Incorporates a-priori information on u
- Sparse $\|u\|_1$, smooth $\|\nabla u\|_2^2$, small $\|u\|_2^2$, etc.

Image Reconstruction

Given (noisy) input data f that is generated by forward model

$$Au = f,$$

recover u using well-known ROF Model Rudin et al. (1992)

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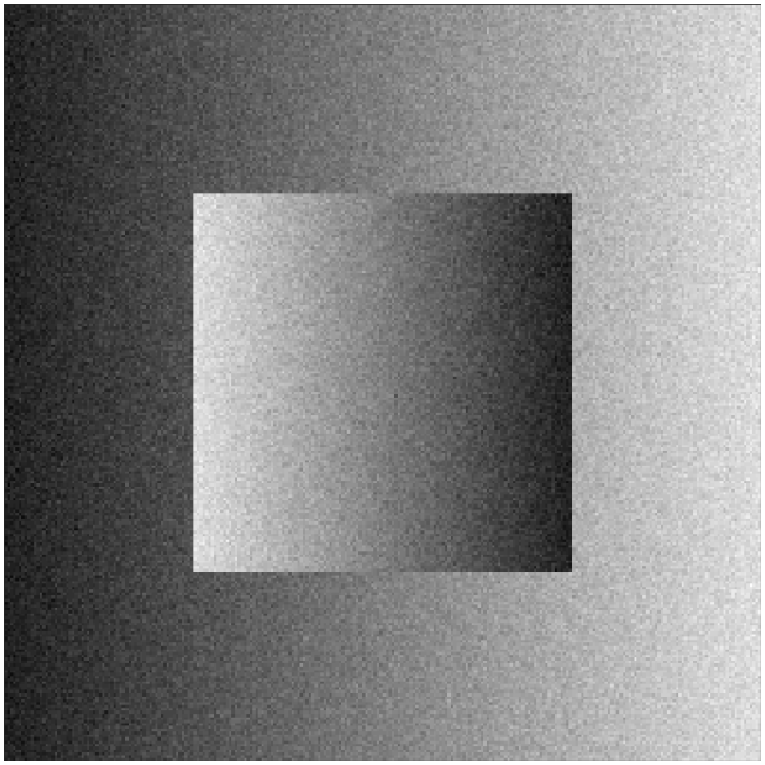
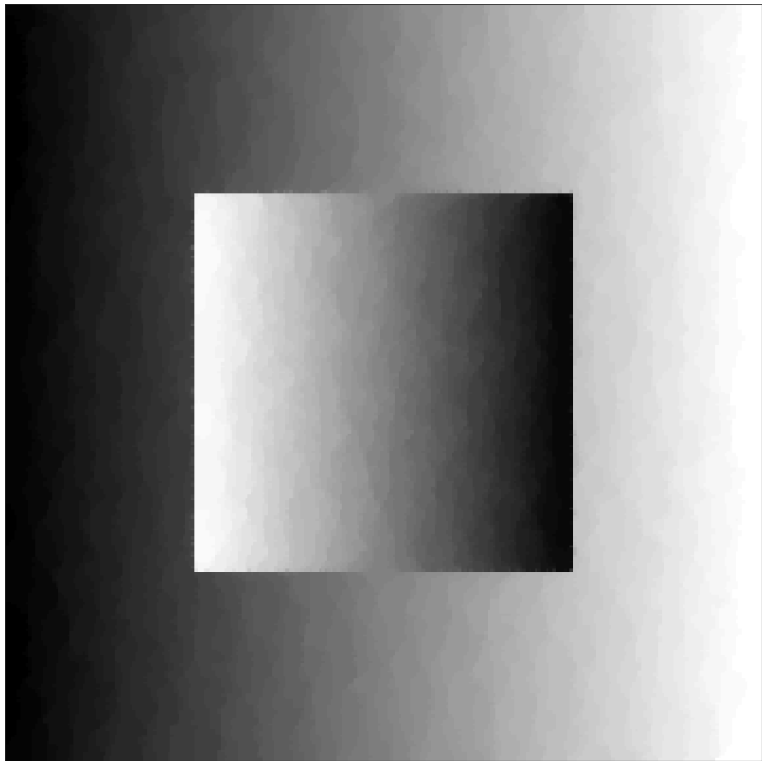
- Linear operator A
- Isotropic total variation:

$$\|\nabla u\|_{2,1} := \sum_{i,j} |\sqrt{(u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2}|$$

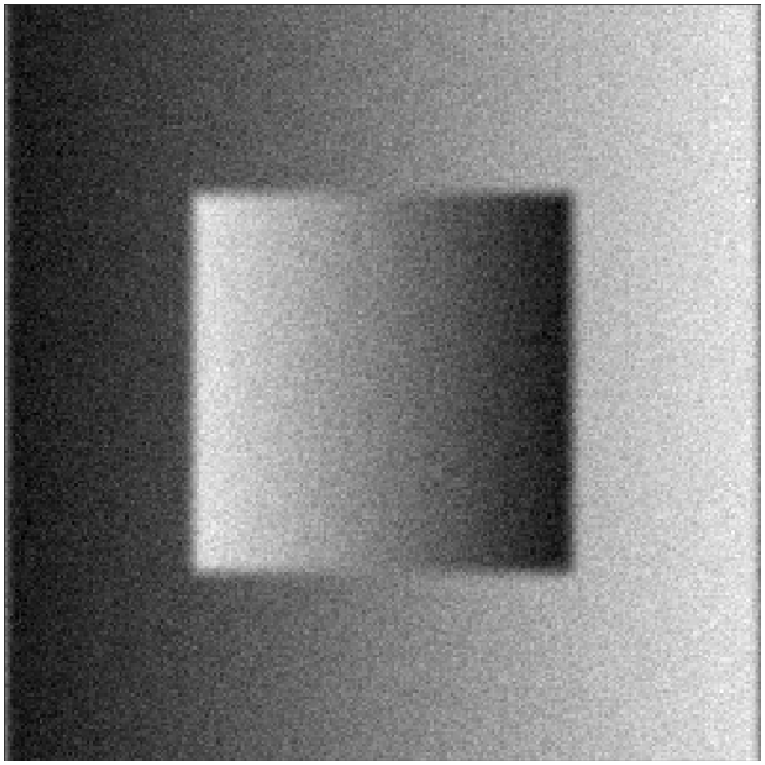
L^2 Denoising

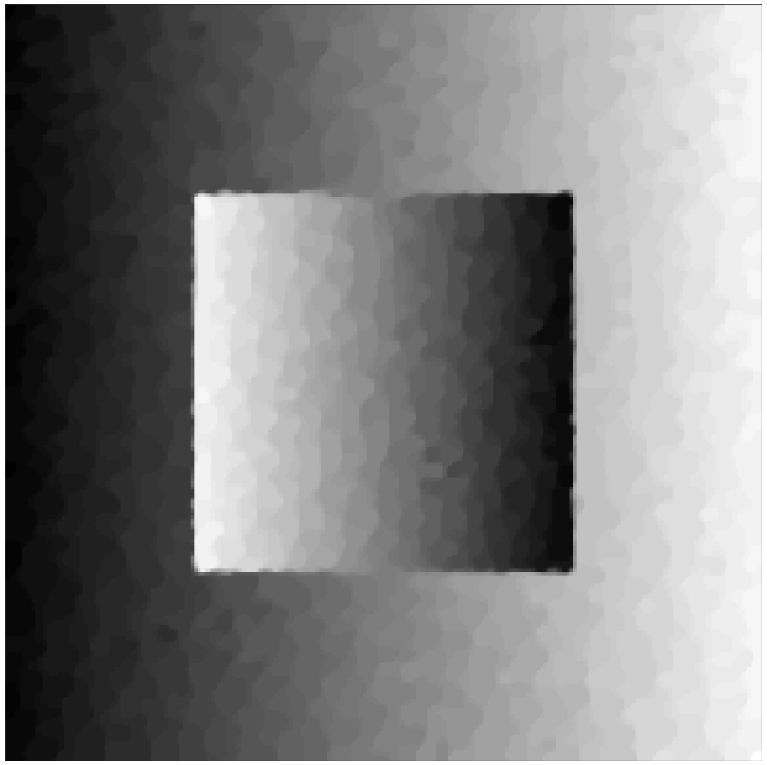




$$\arg\min_u \|u - f^\delta\|_2^2 + \alpha\|\nabla u\|_{2,1}$$

Deblurring





$$\arg\min_u \|g_\sigma \star u - f^\delta\|_2^2 + \alpha\|\nabla u\|_{2,1}$$

Zooming

$$\arg\min_u \|D(g_\sigma \star u) - f^\delta\|_2^2 + \alpha\|\nabla u\|_{2,1}$$

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- Linearization yields **Optical Flow Constraint**

$$0 = I_1 - I_2 + \nabla I_1 \cdot \mathbf{v}$$

- Usefull: Taylor expansion with respect to some a-priori solution $\tilde{\mathbf{v}}$

$$0 = I_1(x + \tilde{\mathbf{v}}) - I_2 + \nabla I_1(x + \tilde{\mathbf{v}}) \cdot (\mathbf{v} + \tilde{\mathbf{v}}) =: \rho(\mathbf{v})$$

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$$\min_{\mathbf{v}} \|\rho(\mathbf{v})\| + \alpha \mathcal{R}(\mathbf{v})$$

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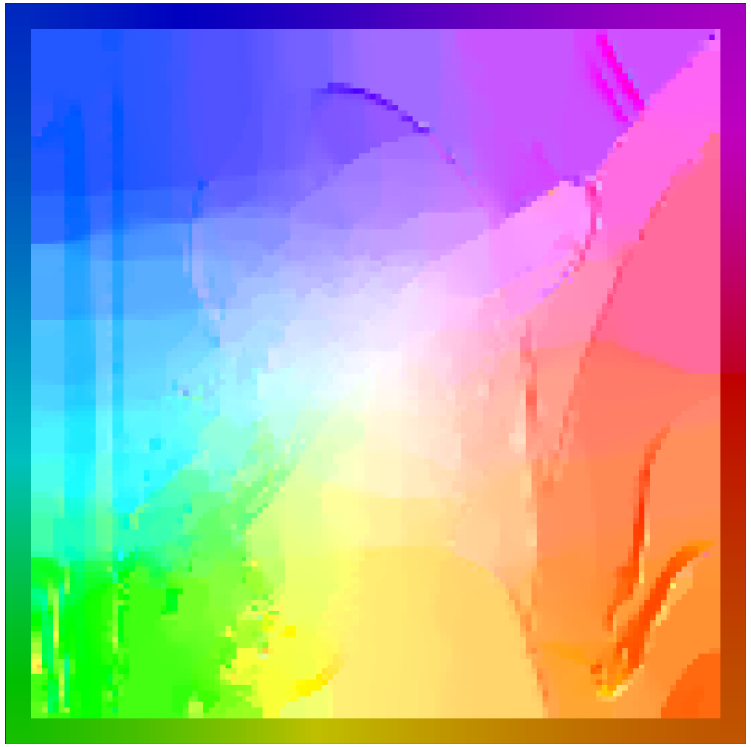
- Optical flow constraint yields one equation per point for two unknowns v_1, v_2 (aperture problem)
- Variational approach takes optical flow constraint as data-fidelity and adds spatial regularity via regularizer
- Wedel et al. (2009):

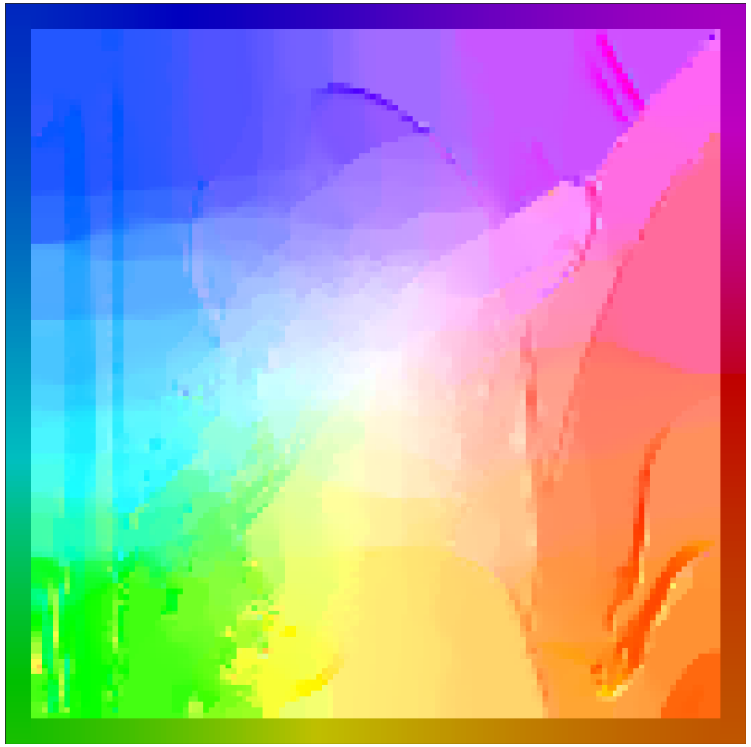
$$\min_{\mathbf{v}} \|\rho(\mathbf{v})\|_1 + \alpha\|\nabla v^1\|_{2,1} + \alpha\|\nabla v^2\|_{2,1}$$

TV-L1 Optical Flow



$$\arg\min_{\mathbf{v}} \|\nabla I_2 \cdot \mathbf{v} + I_2 - I_1\|_2^2 + \alpha\|\nabla v^1\|_{2,1} + \alpha\|\nabla v^2\|_{2,1}$$

TV-L1 Optical Flow



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Optimization:

Primal Formulation

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$$\arg\min_u J(u) = G(u) + F(Ku).$$

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- Part $F(Ku)$ shall contain non-differentiable parts that may incorporate operators

Divide ROF problem

For denoising

$$\arg\min_u \|u - f^\delta\|_2^2 + \alpha\|\nabla u\|_{2,1}$$

we have

$$G(u) = \|u - f^\delta\|_2^2$$
$$F(Ku) = \alpha\|\nabla u\|_{2,1}, K = (D_x, D_y)$$

Splitting

Problem: Part $F(Ku)$ can be non-differentiable and incorporates an operator! Therefore split out the operator and solve

$$\arg\min_{u,d} \mathrm{G}(u) + F(d)$$
$$\text{s.t. } d = Ku$$

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Augmented Lagrangian formulation to minimize the constrained problem. Introduce Lagrange multiplier λ^k

$$(u^{k+1}, d^{k+1}) = \arg\min_{u,d} \ \mathrm{G}(u) + F(d) + \frac{\mu}{2}\|d - Ku\|_2^2 + \langle \lambda^k, d - Ku\rangle$$
$$\lambda^{k+1} = \lambda^k + \mu(d^{k+1} - Ku^{k+1})$$

Modified problem

What has changed?

$$(u^{k+1}, d^{k+1}) = \arg\min_{u,d} \; \mathrm{G}(u) + F(d) + \frac{\mu}{2}\|d - Ku\|_2^2 + \langle \lambda^k, d - Ku \rangle$$

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- Non-differentiable part $F(d)$ does not incorporate operators anymore!

Concatenating u and d into $\tilde{u}$ we can move the right terms into nice part G and from now on solve

$$\arg\min_{\tilde{u}} J(\tilde{u}) = \; \mathrm{G}(\tilde{u}) + F(\tilde{u})$$

Gradient Descent

The direct way to minimize $J(u)$ is a gradient descent:

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for a suitable step length τ^k (Wolfe conditions).

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Problem: Most problems are not (completely) differentiable!
ROF:

- $G(u)$✓
- $F(u)$× due to 1-norm

Resolvent Operator

A tool in convex optimization is the resolvent operator:

$$prox_{\tau G}(\tilde{u}) = (I + \tau \partial G)^{-1}(\tilde{u}) := \arg\min_{u} \left\{ \frac{1}{2} \|u - \tilde{u}\|_2^2 + \tau G(u) \right\}$$

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Important property:
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Leads to fixed point algorithm

$$u^{k+1} = prox_{\tau J}(u^k)$$

Evaluating the prox-problem for J is usually not easier than solving the original problem

FBS

Idea: Split the fixed point algorithm into parts for G and F

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Equivalently

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$\rightarrow$**forward-backward splitting**

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Therefore: Back to the original problem $J(u) = G(u) + F(Ku)$.

- non-differentiable parts involving operators put into $F(Ku)$
- rest $G(u)$

Primal-Dual Problem

Instead of the primal problem we can equivalently solve the primal-dual problem

$$\arg\min_u \arg\max_y G(u) - F^*(y) + \langle Ku, y \rangle$$

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$$\arg\min_u \arg\max_y G(u) - F^*(y) + \langle Ku, y\rangle$$

where

$$F^*(y) = sup_{x\in X}\langle u, y\rangle - J(y)$$

is the so-called *convex conjugate* to F.

- convex
- $F^{**} \leq F$

For our ROF example we had

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$$\begin{aligned} F^*(y) &= \delta_{\{y \in \mathbb{R}^{2N_1N_2} : \|y\|_{2,\infty} \leq \alpha\}}(y) \\ &= \begin{cases} 0 & \text{if } \max_{i,j} \sqrt{(y^1_{i,j})^2 + (y^2_{i,j})^2} \leq \alpha \\ \infty & \text{else} \end{cases} \end{aligned}$$

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Properties of the convex conjugate

- $(F(\cdot + a))^* = F^*(\cdot) - \langle \cdot, a \rangle$,
- $(F(\cdot) + a)^* = F^*(\cdot) - a$
- $(\lambda F(\cdot))^* = \lambda F^*(\frac{\cdot}{\lambda}), \quad \lambda > 0$

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Typical applications

- For $F(u) = \frac{\alpha}{2} \|u\|_2^2$ we have $F^*(y) = \frac{1}{2\alpha} \|y\|_2^2$
- For $F(u) = \|u\|_{\mathcal{X}}$ we have $F^*(y) = \delta_{B(\mathcal{X}^*)}(y)$

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- Alternating steps between y and u
- For y do a gradient ascent for the *nice* part and prox to the non-differentiable part

$$\begin{aligned} y^{k+1} &= prox_{\sigma F^*}\left(y^k + \sigma\nabla_y(G(u^k) + \langle Ku^k, y\rangle)\right) \\ &= prox_{\sigma F^*}(y^k + \sigma Ku^k) \end{aligned}$$

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- For x do a gradient descent for the *nice* part and prox to the non-differentiable part

$$\begin{aligned} u^{k+1} &= prox_{\tau G}\left(u^k - \tau\nabla_u(-F^*(y^{k+1}) + \langle u, K^T y^{k+1}\rangle)\right) \\ &= prox_{\tau G}(u^k - \tau K^T y^{k+1}) \end{aligned}$$

Primal-Dual Algorithm

For suitable $\tau, \sigma > 0$ the primal-dual problem can be solved by the following algorithm

$$
\begin{aligned}
y^{k+1} &= prox_{\sigma F^*}(y^k + \sigma K \bar{u}^k) \\
u^{k+1} &= prox_{\tau G}(u^k - \tau K^T y^{k+1}) \\
\bar{u}^{k+1} &= 2u^{k+1} - u^k
\end{aligned}
$$

Proposed by Chambolle and Pock in 2010 [Chambolle and Pock (2011)], earlier mentioned by Pock, Cremers, Bischof, Chambolle in Pock et al. (2009).

Applied to the ROF problem:

$$\arg\min_u \left\{ \frac{1}{2} \|u - \tilde{u}\|_2^2 + \frac{\tau}{2} \|u - f^\delta\|_2^2 \right\}$$

Optimality reads:

$$0 = u - \tilde{u} + \tau(u - f^\delta) \Leftrightarrow u = \frac{\tilde{u} + f^\delta}{1 + \tau}$$

$$prox_{\sigma F^*}(\tilde{y}) = \arg\min_y \left\{ \frac{1}{2} \|y - \tilde{y}\|_2^2 + \sigma F^*(y) \right\}$$
$$= \arg\min_y \left\{ \frac{1}{2} \|y - \tilde{y}\|_2^2 + \sigma \delta_{\{y \in \mathbb{R}^{2N_1N_2} : \|y\|_{2,\infty} \leq \alpha\}}(y) \right\}$$

Therefore, the solution is given by a componentwise projection onto l^2-unit balls

$$y_{i,j}^{1,2} = \begin{cases} \tilde{y}_{i,j}^{1,2} & \text{if } \sqrt{(\tilde{y}_{i,j}^1)^2 + (\tilde{y}_{i,j}^2)^2} \leq \alpha \\ \alpha \frac{\tilde{y}_{i,j}^{1,2}}{\sqrt{(\tilde{y}_{i,j}^1)^2 + (\tilde{y}_{i,j}^2)^2}} & \text{else} \end{cases}$$

Short vectorial notation: $y = \pi_{\alpha \cdot l^2}(\tilde{y})$

Primal-Dual Algorithm for ROF Problem

Inserting the prox formulae for G and F leads to the algorithm

$$
\begin{aligned}
y^{k+1} &= \pi_{\alpha \cdot l^2}(y^k + \sigma K \bar{u}^k) \\
u^{k+1} &= \frac{u^k - \tau K^T y^{k+1} + f^\delta}{1 + \tau} \\
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\end{aligned}
$$

Stopping criterion

Direct way:

- Measure distance $d = \|u^{k+1} - u^k\| + \|y^{k+1} - y^k\|$
- Stop if $d < \epsilon$
- Problem: Convergence not guaranteed
- Distance to minimizer may be arbitrarily large

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Duality gap

- $g(u, y) = G(x) + F(Kx) + G^*(-K^T y) + F^*(y)$
- Upper bound for distance to minimal value of functional
- stop if $g(u^k, y^k) < \epsilon$

FlexBox:

General Setup

http://www.flexbox.im

- Toolbox that aims at solving typical convex problems in our field

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- Implements primal-dual minimization algorithm in a general setting
- Convexity required (formally)
- Problem may be non-differentiable and/or involve linear operators
- Toolbox takes your variational problem as list of terms

ROF MATLAB example

Features

- Combine arbitrary terms

Features

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- Automated optimal stepsize calculation

Features

- Combine arbitrary terms
- Automated optimal stepsize calculation
- Automated stopping criterion

Features

- Combine arbitrary terms
- Automated optimal stepsize calculation
- Automated stopping criterion
- Standalone C++ and CUDA module for massive acceleration

Operators

Preimplemented Operators

- Gradient

Operators

Preimplemented Operators

- Gradient
- Convolution

Operators

Preimplemented Operators

- Gradient
- Convolution
- Diagonal, Identity, Zero

Operators

Preimplemented Operators

- Gradient
- Convolution
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Operators

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- Function handle

Available terms

Data fidelity for reconstruction

- $\alpha \|Au - f\|_1$

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- $\alpha \|Au - f\|_1$
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- $Au - f + f \log \frac{f}{Au}$

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- $\alpha \|Au\|_{H_\epsilon}$

Available terms

Vector Regularization:

- $\alpha \| curl(\mathbf{v}) \|_1$

Available terms

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- $\alpha \| \nabla \cdot \mathbf{v} \|_1$

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Available terms

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Many

- more . . .

Further MATLAB demos

Thank you!

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