Convex Optimization in Imaging
CoSIP Winter Retreat 2016
Examples from Image Processing
  Image Reconstruction
  Optical Flow

Optimization
  Operator Splitting
  Forward-Backward-Splitting
  Primal-Dual-Method

FlexBox
Examples from Image Processing:
General Consideration

- Finite dimensional optimization problem in $\mathbb{R}^d$
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- For images we usually have rectangular grid of size $N_1 \times N_2$ and search for $u \in \mathbb{R}^{N_1 \times N_2}$
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- For images we usually have rectangular grid of size $N_1 \times N_2$ and search for $u \in \mathbb{R}^{N_1 \times N_2}$
- Concatenating columns yields vector $u \in \mathcal{X} = \mathbb{R}^{N_1N_2}$
Variational Model

Introduced problems can be expressed as general Variational Problem

\[
\arg\min_{u} D(u, f) + \alpha R(u)
\]
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Data fidelity:
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- $D(u, f)$ arises from the direct model for the inverse problem
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Regularization:
- Incorporates a-priori information on $u$
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- May incorporate linear operator $A$
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Regularization:

- Incorporates a-priori information on $u$
- Sparse $\|u\|_1$, smooth $\|\nabla u\|_2^2$, small $\|u\|_2^2$, etc.
Image Reconstruction

Given (noisy) input data $f$ that is generated by forward model

$$Au = f,$$

recover $u$ using well-known ROF Model Rudin et al. (1992)
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$$\arg\min_u \|Au - f\|^2_2 + \alpha \|\nabla u\|_{2,1}$$
Image Reconstruction

Given (noisy) input data $f$ that is generated by forward model $Au = f$, recover $u$ using well-known ROF Model Rudin et al. (1992)

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- Linear operator $A$
Image Reconstruction

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recover $u$ using well-known ROF Model Rudin et al. (1992)

$$\arg\min_u \|Au - f\|_2^2 + \alpha \|\nabla u\|_{2,1}$$

- Linear operator $A$
- Isotropic total variation:

$$\|\nabla u\|_{2,1} := \sum_{i,j} \left| \sqrt{(u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2} \right|$$
\( L^2 \) Denoising

\[
\arg \min_u \| u - f^\delta \|^2_2 + \alpha \| \nabla u \|_{2,1}
\]
Deblurring

\[
\arg \min_u \left\| g_\sigma \ast u - f^\delta \right\|_2^2 + \alpha \| \nabla u \|_{2,1}
\]
Zooming

\[
\arg\min_u \|D(g_\sigma \ast u) - f^\delta\|_2^2 + \alpha \|\nabla u\|_{2,1}
\]
Regularization

- $\|\nabla u\|_2^2$ smoothness
Regularization

- $\|\nabla u\|_2^2$ smoothness
- $\|\nabla u\|_{1,2}$ piecewise constant
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- Higher order TV
Regularization

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- $\|\nabla u\|_{1,2}$ piecewise constant
- Higher order TV
- Infimal convolution of several regularizers
Optical Flow

Problem

- Given two images $I_1, I_2$ showing the same scene at different time
Optical Flow

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- Given two images $I_1, I_2$ showing the same scene at different time
- Want to estimate velocity field $\mathbf{v} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ describing the displacement between both images
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- Common assumption: Brightness constancy

\[ I_1(x + v) = I_2(x) \]
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  $$I_1(x + v) = I_2(x)$$
- Linearization yields **Optical Flow Constraint**
  $$0 = I_1 - I_2 + \nabla I_1 \cdot v$$
Optical Flow

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- Given two images $I_1, I_2$ showing the same scene at different time
- Want to estimate velocity field $\mathbf{v} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ describing the displacement between both images
- Common assumption: Brightness constancy
  \[ I_1(x + \mathbf{v}) = I_2(x) \]
- Linearization yields Optical Flow Constraint
  \[ \mathbf{0} = I_1 - I_2 + \nabla I_1 \cdot \mathbf{v} \]
- Usefull: Taylor expansion with respect to some a-priori solution $\tilde{\mathbf{v}}$
  \[ \mathbf{0} = I_1(x + \tilde{\mathbf{v}}) - I_2 + \nabla I_1(x + \tilde{\mathbf{v}}) \cdot (\mathbf{v} + \tilde{\mathbf{v}}) =: \rho(\mathbf{v}) \]
Variational Approach

Problem:

- Optical flow constraint yields one equation per point for two unknowns $v_1, v_2$ (aperture problem)
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- Variational approach takes optical flow constraint as data-fidelity and adds spatial regularity via regularizer

$$\min_{\mathbf{v}} \| \rho(\mathbf{v}) \| + \alpha \mathcal{R}(\mathbf{v})$$
Variational Approach

Problem:

- Optical flow constraint yields one equation per point for two unknowns $v_1, v_2$ (aperture problem)
- Variational approach takes optical flow constraint as data-fidelity and adds spatial regularity via regularizer
- Wedel et al. (2009):

$$\min_v \| \rho(v) \|_1 + \alpha \| \nabla v^1 \|_{2,1} + \alpha \| \nabla v^2 \|_{2,1}$$
TV-L1 Optical Flow

$$\arg \min \| \nabla l_2 \cdot v + l_2 - l_1 \|_2^2 + \alpha \| \nabla v^1 \|_{2,1} + \alpha \| \nabla v^2 \|_{2,1}$$
TV-L1 Optical Flow

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\arg\min_v \| \nabla l_2 \cdot v + l_2 - l_1 \|_2^2 + \alpha \| \nabla v^1 \|_{2,1} + \alpha \| \nabla v^2 \|_{2,1}
\]
Optimization:
Primal Formulation

The introduced variational problems can be expressed as

$$\arg\min_{u} J(u) = G(u) + F(Ku).$$
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$$\arg \min_u J(u) = G(u) + F(Ku).$$

- Part $G(u)$ shall contain differentiable parts (even linear operators)
- Part $F(Ku)$ shall contain non-differentiable parts that may incorporate operators
Divide ROF problem

For denoising

$$\arg\min_u \|u - f^\delta\|_2^2 + \alpha \|\nabla u\|_{2,1}$$

we have

$$G(u) = \|u - f^\delta\|_2^2$$
$$F(Ku) = \alpha \|\nabla u\|_{2,1}, \quad K = (D_x, D_y)$$
Splitting

Problem: Part $F(Ku)$ can be non-differentiable and incorporates an operator! Therefore split out the operator and solve

$$\arg \min_{u,d} G(u) + F(d)$$

s.t. $d = Ku$
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s.t. $d = Ku$

Augmented Lagrangian formulation to minimize the constrained problem. Introduce Lagrange multiplier $\lambda^k$

$$(u^{k+1}, d^{k+1}) = \arg \min_{u,d} G(u) + F(d) + \frac{\mu}{2} \|d - Ku\|^2 + \langle \lambda^k, d - Ku \rangle$$

$$\lambda^{k+1} = \lambda^k + \mu (d^{k+1} - Ku^{k+1})$$
Modified problem

What has changed?

\[(u^{k+1}, d^{k+1}) = \arg\min_{u,d} G(u) + F(d) + \frac{\mu}{2} \|d - Ku\|^2 + \langle \lambda^k, d - Ku \rangle\]

- Non-differentiable part \( F(d) \) does not incorporate operators anymore!
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\[(u^{k+1}, d^{k+1}) = \arg\min_{u,d} G(u) + F(d) + \frac{\mu}{2} \|d - Ku\|_2^2 + \langle \lambda^k, d - Ku \rangle\]

- Non-differentiable part \(F(d)\) does not incorporate operators anymore!

Concatenating \(u\) and \(d\) into \(\tilde{u}\) we can move the right terms into nice part \(G\) and from now on solve

\[\arg\min_{\tilde{u}} J(\tilde{u}) = G(\tilde{u}) + F(\tilde{u})\]
Gradient Descent

The direct way to minimize $J(u)$ is a gradient descent:

$$u^{k+1} = u^k - \tau^k \nabla J(u^k)$$

for a suitable step length $\tau^k$ (Wolfe conditions).
Gradient Descent

The direct way to minimize $J(u)$ is a gradient descent:

$$u^{k+1} = u^k - \tau^k \nabla J(u^k)$$

for a suitable step length $\tau^k$ (Wolfe conditions).

Problem: Most problems are not (completely) differentiable!

ROF:

- $G(u)\checkmark$
- $F(u)\times$ due to 1-norm
Resolvent Operator

A tool in convex optimization is the resolvent operator:

\[
\text{prox}_{\tau G}(\tilde{u}) = (I + \tau \partial G)^{-1}(\tilde{u}) := \arg \min_u \left\{ \frac{1}{2} \|u - \tilde{u}\|_2^2 + \tau G(u) \right\}
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Resolvent Operator

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Important property:
For convex functional $J$ and arbitrary $\tau > 0$ the element $\bar{u} \in \text{dom}(J)$ is minimizer iff

$$\bar{u} = \text{prox}_{\tau J}(\tilde{u})$$
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\[
\bar{u} = \text{prox}_{\tau J}(\bar{u})
\]

Leads to fixed point algorithm

\[
u^{k+1} = \text{prox}_{\tau J}(u^k)
\]

Evaluating the prox-problem for \( J \) is usually not easier than solving the original problem
Idea: Split the fixed point algorithm into parts for $G$ and $F$

$$u^{k+\frac{1}{2}} = u^k - \gamma \nabla G(u^k)$$
**Idea:** Split the fixed point algorithm into parts for $G$ and $F$

\[
\begin{align*}
    u^{k+\frac{1}{2}} &= u^k - \gamma \nabla G(u^k) \\
    u^{k+1} &= \text{prox}_{\gamma F}(u^{k+\frac{1}{2}}) = \arg\min_u \frac{1}{2} \| u - u^{k+\frac{1}{2}} \|^2 + \gamma F(u)
\end{align*}
\]
FBS

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Equivalently

\[
u^{k+1} = \text{prox}_{\gamma F}(u^k - \gamma \nabla G(u^k))
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- Gradient descent for *nice* part $G$
- prox operation for non-differentiable part
**FBS**

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    u^{k+1} = \text{prox}_{\gamma F}(u^k - \gamma \nabla G(u^k))
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- Gradient descent for *nice* part $G$
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→ forward-backward splitting
Problems:
- Operator decoupling
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- FBS just sub-problem (alternating between update of Lagrange multiplier and FBS)
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- Operator decoupling
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- Step-length $\gamma$ (line-search)

Therefore: Back to the original problem $J(u) = G(u) + F(Ku)$.

- non-differentiable parts involving operators put into $F(Ku)$
- rest $G(u)$
Primal-Dual Problem

Instead of the primal problem we can equivalently solve the primal-dual problem

$$\arg \min_u \arg \max_y G(u) - F^*(y) + \langle Ku, y \rangle$$
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$$\arg \min_u \arg \max_y G(u) - F^*(y) + \langle Ku, y \rangle$$

where

$$F^*(y) = \sup_{x \in X} \langle u, y \rangle - J(y)$$

is the so-called *convex conjugate* to F.

- convex
- $F^{**} \leq F$
For our ROF example we had

\[ F(Ku) = \alpha \|\nabla u\|_{2,1}, \quad K = (D_x, D_y) \]
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and calculate

\[ F^*(y) = \delta_{\{y \in \mathbb{R}^{2N_1N_2} : \|y\|_{2,\infty} \leq \alpha\}}(y) = \begin{cases} 0 & \text{if } \max_{i,j} \sqrt{(y_{1,i,j})^2 + (y_{2,i,j})^2} \leq \alpha \\ \infty & \text{else} \end{cases} \]
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0 & \text{if } \max_{i,j} \sqrt{(y_{i,j}^1)^2 + (y_{i,j}^2)^2} \leq \alpha \\
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\end{cases} \]

Leads to the primal-dual problem

\[ \arg \min_u \arg \max_y \|u - f^\delta\|_2^2 - \delta_{\{y \in \mathbb{R}^{2N_1N_2} : \|y\|_{2,\infty} \leq \alpha\}}(y) + \langle Ku, y \rangle \]
Properties of the convex conjugate

\( (F(\cdot + a))^* = F^*(\cdot) - \langle \cdot, a \rangle , \)
\( (F(\cdot) + a)^* = F^*(\cdot) - a \)
\( (\lambda F(\cdot))^* = \lambda F^*(\frac{\cdot}{\lambda}), \quad \lambda > 0 \)
Properties of the convex conjugate

- \((F(\cdot + a))^* = F^*(\cdot) - \langle \cdot, a \rangle\),
- \((F(\cdot) + a)^* = F^*(\cdot) - a\)
- \((\lambda F(\cdot))^* = \lambda F^*(\frac{\cdot}{\lambda}), \quad \lambda > 0\)

Typical applications

- For \(F(u) = \frac{\alpha}{2} \|u\|_2^2\) we have \(F^*(y) = \frac{1}{2\alpha} \|y\|_2^2\)
- For \(F(u) = \|u\|_\chi\) we have \(F^*(y) = \delta_B(\chi^*)(y)\)
Idea: For the primal-dual problem

$$\arg \min_u \arg \max_y G(u) - F^*(y) + \langle Ku, y \rangle$$
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- Alternating steps between $y$ and $u$
Idea: For the primal-dual problem

$$\arg \min_u \arg \max_y G(u) - F^*(y) + \langle Ku, y \rangle$$

- Alternating steps between $y$ and $u$
- For $y$ do a gradient ascent for the *nice* part and prox to the non-differentiable part

$$y^{k+1} = \text{prox}_{\sigma F^*} \left( y^k + \sigma \nabla_y (G(u^k) + \langle Ku^k, y \rangle) \right)$$

$$= \text{prox}_{\sigma F^*} (y^k + \sigma Ku^k)$$
Idea: For the primal-dual problem

$$\arg\min_u \arg\max_y G(u) - F^*(y) + \langle Ku, y \rangle$$

- Alternating steps between $y$ and $u$
- For $y$ do a gradient ascent for the nice part and prox to the non-differentiable part
  $$y^{k+1} = \text{prox}_{\sigma F^*} \left( y^k + \sigma \nabla_y (G(u^k) + \langle Ku^k, y \rangle) \right)$$
  $$= \text{prox}_{\sigma F^*}(y^k + \sigma Ku^k)$$
- For $u$ do a gradient descent for the nice part and prox to the non-differentiable part
  $$u^{k+1} = \text{prox}_{\tau G} \left( u^k - \tau \nabla_u (-F^*(y^{k+1}) + \langle u, K^T y^{k+1} \rangle) \right)$$
  $$= \text{prox}_{\tau G}(u^k - \tau K^T y^{k+1})$$
Primal-Dual Algorithm

For suitable $\tau, \sigma > 0$ the primal-dual problem can be solved by the following algorithm

\[
\begin{align*}
y^{k+1} &= \text{prox}_{\sigma F^*}(y^k + \sigma K\bar{u}^k) \\
u^{k+1} &= \text{prox}_{\tau G}(u^k - \tau K^T y^{k+1}) \\
\bar{u}^{k+1} &= 2u^{k+1} - u^k
\end{align*}
\]

Proposed by Chambolle and Pock in 2010 [Chambolle and Pock (2011)], earlier mentioned by Pock, Cremers, Bischof, Chambolle in Pock et al. (2009).
Applied to the ROF problem:

\[
\text{arg min}_{u} \left\{ \frac{1}{2} \| u - \tilde{u} \|_2^2 + \frac{\tau}{2} \| u - f^\delta \|_2^2 \right\}
\]

Optimality reads:

\[
o = u - \tilde{u} + \tau (u - f^\delta) \iff u = \frac{\tilde{u} + f^\delta}{1 + \tau}
\]
\[
prox_{\sigma F^*}(\tilde{y}) = \arg \min_y \left\{ \frac{1}{2} \|y - \tilde{y}\|_2^2 + \sigma F^*(y) \right\}
\]

\[
= \arg \min_y \left\{ \frac{1}{2} \|y - \tilde{y}\|_2^2 + \sigma \delta_{\{y \in \mathbb{R}^{2N_1N_2} : \|y\|_{2,\infty} \leq \alpha\}}(y) \right\}
\]

Therefore, the solution is given by a componentwise projection onto \(l^2\)-unit balls:

\[
y_{1,2}^{i,j} = \begin{cases} 
\tilde{y}_{1,2}^{i,j} & \text{if } \sqrt{(\tilde{y}_{1}^{i,j})^2 + (\tilde{y}_{2}^{i,j})^2} \leq \alpha \\
\frac{\alpha \tilde{y}_{1,2}^{i,j}}{\sqrt{(\tilde{y}_{1}^{i,j})^2 + (\tilde{y}_{2}^{i,j})^2}} & \text{else}
\end{cases}
\]

Short vectorial notation: \(y = \pi_{\alpha \cdot l^2}(\tilde{y})\)
Inserting the prox formulae for $G$ and $F$ leads to the algorithm

\[
y^{k+1} = \pi_{\alpha \cdot l^2} (y^k + \sigma K \tilde{u}^k)
\]
\[
u^{k+1} = \frac{u^k - \tau K^T y^{k+1} + f^\delta}{1 + \tau}
\]
\[
\bar{u}^{k+1} = 2u^{k+1} - u^k
\]
Stopping criterion

Direct way:
- Measure distance \( d = \| u^{k+1} - u^k \| + \| y^{k+1} - y^k \| \)
- Stop if \( d < \epsilon \)
- Problem: Convergence not guaranteed
- Distance to minimizer may be arbitrarily large
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Direct way:

- Measure distance \( d = \|u^{k+1} - u^k\| + \|y^{k+1} - y^k\| \)
- Stop if \( d < \epsilon \)
- Problem: Convergence not guaranteed
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Duality gap

- \( g(u, y) = G(x) + F(Kx) + G^*(-K^Ty) + F^*(y) \)
- Upper bound for distance to minimal value of functional
- Stop if \( g(u^k, y^k) < \epsilon \)
FlexBox:
General Setup

http://www.flexbox.im

- Toolbox that aims at solving typical convex problems in our field
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- Basic framework written in MATLAB
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- Toolbox that aims at solving typical convex problems in our field
- Basic framework written in MATLAB
- Implements primal-dual minimization algorithm in a general setting
- Convexity required (formally)
- Problem may be non-differentiable and/or involve linear operators
- Toolbox takes your variational problem as list of terms
ROF MATLAB example
Features

- Combine arbitrary terms
Features

- Combine arbitrary terms
- Automated optimal stepsize calculation
Features

- Combine arbitrary terms
- Automated optimal stepsize calculation
- Automated stopping criterion
Features

- Combine arbitrary terms
- Automated optimal stepsize calculation
- Automated stopping criterion
- Standalone C++ and CUDA module for massive acceleration
Operators

Preimplemented Operators

- Gradient
Operators

Preimplemented Operators

- Gradient
- Convolution
Operators

Preimplemented Operators

- Gradient
- Convolution
- Diagonal, Identity, Zero
Operators

Preimplemented Operators

- Gradient
- Convolution
- Diagonal, Identity, Zero
- Downsampling
Operators

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- Gradient
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- Concatenation
Operators

Preimplemented Operators

- Gradient
- Convolution
- Diagonal, Identity, Zero
- Downsampling
- Concatenation
- Function handle
Available terms

Data fidelity for reconstruction

- \( \alpha \| Au - f \|_1 \)
Available terms

Data fidelity for reconstruction

- $\alpha \|Au - f\|_1$
- $\frac{\alpha}{2} \|Au - f\|_2^2$
Available terms

Data fidelity for reconstruction

\[ \alpha \| Au - f \|_1 \]
\[ \frac{\alpha}{2} \| Au - f \|_2^2 \]
\[ Au - f + f \log \frac{f}{Au} \]
Available terms

Data fidelity for reconstruction

- $\alpha \| Au - f \|_1$
- $\frac{\alpha}{2} \| Au - f \|_2^2$
- $Au - f + f \log \frac{f}{Au}$

Data fidelity for optical flow

- $\alpha \| \nabla f_2 \cdot v + f_2 - f_1 \|_2^2$
Available terms

Data fidelity for reconstruction
- $\alpha \|Au - f\|_1$
- $\frac{\alpha}{2} \|Au - f\|_2^2$
- $Au - f + f \log \frac{f}{Au}$

Data fidelity for optical flow
- $\alpha \|\nabla f_2 \cdot \mathbf{v} + f_2 - f_1\|_2^2$
- $\alpha \|\nabla f_2 \cdot \mathbf{v} + f_2 - f_1\|_1$
Available terms

Data fidelity for reconstruction
- $\alpha \| Au - f \|_1$
- $\frac{\alpha}{2} \| Au - f \|_2^2$
- $Au - f + f \log \frac{f}{Au}$

Data fidelity for optical flow
- $\alpha \| \nabla f_2 \cdot \mathbf{v} + f_2 - f_1 \|_2^2$
- $\alpha \| \nabla f_2 \cdot \mathbf{v} + f_2 - f_1 \|_1$

Regularization
- $\alpha \| Au \|_{1,2}$
Available terms

Data fidelity for reconstruction

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Data fidelity for optical flow

- $\alpha \|\nabla f_2 \cdot \mathbf{v} + f_2 - f_1\|_2^2$
- $\alpha \|\nabla f_2 \cdot \mathbf{v} + f_2 - f_1\|_1$

Regularization

- $\alpha \|Au\|_{1,2}$
- $\frac{\alpha}{2} \|Au\|_2^2$
- $\alpha \|Au\|_{H^\epsilon}$
Available terms

Vector Regularization:

- $\alpha \| \text{curl}(v) \|_1$
Available terms

Vector Regularization:

- $\alpha \| \text{curl}(\mathbf{v}) \|_1$
- $\alpha \| \nabla \cdot \mathbf{v} \|_1$
Available terms

Vector Regularization:

- $\alpha \|\text{curl}(\mathbf{v})\|_1$
- $\alpha \|\nabla \cdot \mathbf{v}\|_1$

Other Regularization:

- $\alpha \langle \mathbf{b}, \nabla \mathbf{u} \rangle$
Available terms

Vector Regularization:
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- $\frac{\alpha}{2} \| \mathbf{u} \|_2^2$
Available terms

Vector Regularization:

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▶ $\alpha \| \nabla \cdot \mathbf{v} \|_1$

Other Regularization:

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▶ $\alpha \| u \|_1$
▶ $\alpha \| u \|_2$
▶ $\frac{\alpha}{2} \| u \|_2^2$

Many
▶ more . . .
Further MATLAB demos
Thank you!
References I


References II


References III