Interpolatory Model Reduction for Flow Control

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Outline

- Flow Control Problem: Wake Stabilization by Cylinder Rotation
  - Methodology and the LQR Problem
  - Discretization and Linearization
- Model Reduction Problem For DAEs
  - Model Reduction by Interpolation
  - Interpolation for Oseen Equations
  - Numerical Results
- Nonlinear Models
  - Bilinear Systems
  - Quadratic-in-state Systems
- Conclusions and Future Work
Wake Stabilization by Cylinder Rotation

Objective
Stabilize the wake behind a circular cylinder using cylinder rotation.

Plan
Use linear feedback control to stabilize the steady-state solution.

Figure: Steady-State Velocity Components at $Re_d = 60$
Linearize about the steady-state

- An incomplete list: [Tokumaru/Dimotakis, 91], [Blackburn/Henderson, 99], [Dennis et al., 00], [He et al., 00], [Bergmann et al., 00], [Noack et al., 03], [Gerhard et al., 03], [Stoyanov, 09], [Benner/Heiland, 14], ...

- Linearize the Navier-Stokes equations about the steady-state flow:

\[
\begin{align*}
v(t) &= \mathbf{V} + \mathbf{v}'(t) \\
p(t) &= \mathbf{P} + \mathbf{p}'(t), \quad t > 0.
\end{align*}
\]

- Leads to the Oseen Equations

\[
\begin{align*}
\mathbf{v}'_t &= -\mathbf{V} \cdot \nabla \mathbf{v}' - \mathbf{v}' \cdot \nabla \mathbf{V} + \tau(\mathbf{v}') - \nabla \mathbf{p}' + Bu \\
0 &= \nabla \cdot \mathbf{v}'
\end{align*}
\]

where \( \tau(\mathbf{v}) \equiv \mu \left( \nabla \mathbf{v} + \nabla \mathbf{v}^T \right) \), with boundary conditions

- Inflow: \( \mathbf{v}'(t) = 0, \quad t > 0. \)
- Outflow edges: \( (\mathbf{v}'(t), \mathbf{p}'(t)) \) is stress-free.
- \( Bu(t) \) provides tangential velocity on the cylinder.
Two-cylinder case

Controlled Outputs:

\[ y_{i*}(t) = \int_{\Omega_i} v'_j(\xi, t) d\xi \]

where \( i = 1, \ldots, 6 \) and \( j = 1, 2 \).

LQR Problem:
Find \( u(\cdot) \) (tangential velocities) that minimizes

\[ J(u(\cdot)) = \int_0^\infty y^T(t)y(t) + 10\|u(t)\|^2 dt. \]

Seek feedback solutions in the form

\[ u_i(t) = -\int_{\Omega} h^i_1(\xi)v'_1(t, \xi) + h^i_2(\xi)v'_2(t, \xi) dt. \]
Computation of $h_1^i(\cdot), h_2^i(\cdot)$

- Solve steady-state Navier-Stokes equations for $\mathcal{V}$.
- Discretized Oseen equations and control outputs

$$E \dot{x} = Ax + Bu \quad y = Cx$$

where

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$$

- $E_{11} \in \mathbb{R}^{n_1 \times n_1}$ has full rank.
- $A_{11} \in \mathbb{R}^{n_1 \times n_1}, A_{21} \in \mathbb{R}^{n_2 \times n_1}, B_1 \in \mathbb{R}^{n_1 \times 2}$ and $C_1 \in \mathbb{R}^{12 \times n_1}$.
- $A_{21}$ has full rank and $A_{21} E_{11}^{-1} A_{21}^T$ is nonsingular.
The LQR problem becomes: Find a control $u(\cdot)$ that solves

$$\min_u \int_0^\infty \left\{ x_1^T(t)C_1^T C_1 x_1(t) + 10\|u\|^2(t) \right\} \, dt,$$

subject to

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t),$$

$$u(t) = -Kx(t)$$

Computing $K$ requires solving an $n = n_1 + n_2$ dimensional large-scale algebraic Riccati equation:

Instead, reduce the dimension first.
Apply **Interpolatory Model Reduction** to obtain

\[
\tilde{E} \dot{\tilde{x}} = \tilde{A} \tilde{x} + \tilde{B} u(t) \\
\tilde{y} = \tilde{C} \tilde{x}
\]

where $\tilde{E} \in \mathbb{R}^{r \times r}$, $\tilde{A} \in \mathbb{R}^{r \times r}$, $\tilde{B} \in \mathbb{R}^{r \times 2}$, and $\tilde{C} \in \mathbb{R}^{12 \times r}$ with $r \ll n = n_1 + n_2$

Solve the reduced LQR problem

\[
\tilde{A}_{11}^T \tilde{P} \tilde{E}_{11} + \tilde{E}_{11}^T \tilde{P} \tilde{A}_{11} - \tilde{E}_{11}^T \tilde{P} \tilde{B}_1 R^{-1} \tilde{B}_1^T \tilde{P} \tilde{E}_{11} + \tilde{C}_1^T \tilde{C}_1 = 0 \\
\tilde{K} = R^{-1} \tilde{B}_1^T \tilde{P} \tilde{E}_{11}.
\]

Then

\[
u = - \tilde{K} \tilde{x} \\
= - \begin{pmatrix} \tilde{K}V^T \\ V \tilde{x} \end{pmatrix} \\
\approx K \approx x
\]
Interpolatory Model Reduction for DAEs

- Full-order model: Linearized/Discretized Model

\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &=Cx(t) + Du(t),
\end{align*}
\]

- \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n},\) and \(D \in \mathbb{R}^{p \times m}.\)

- Let \(U(s)\) and \(Y(s)\) denote the Laplace transforms of \(u(t)\) and \(y(t)\)

- Transfer function:

\[
Y(s) = G(s)U(s), \quad \text{where} \quad G(s) = C(sE - A)^{-1}B + D.
\]
Model Reduction

The goal is to construct a reduced model of the form
\[
\tilde{E} \dot{\tilde{x}}(t) = \tilde{A} \tilde{x}(t) + \tilde{B} u(t), \quad \tilde{y}(t) = \tilde{C} \tilde{x}(t) + \tilde{D} u(t),
\]
where \( \tilde{E}, \tilde{A} \in \mathbb{R}^{r \times r}, \tilde{B} \in \mathbb{R}^{r \times m}, \tilde{C} \in \mathbb{R}^{p \times r}, \) and \( \tilde{D} \in \mathbb{R}^{p \times m} \) with \( r \ll n \).

Construct \( V \in \mathbb{R}^{n \times r} \) and \( W^T \in \mathbb{R}^{n \times r} \), assume \( x(t) \approx V \tilde{x}(t) \):
\[
\tilde{E} = W^T E V, \quad \tilde{A} = W^T A V, \quad \tilde{B} = W^T B, \quad \text{and} \quad \tilde{C} = C V.
\]

Define \( \tilde{G}(s) = C (s \tilde{E} - \tilde{A})^{-1} \tilde{B} + \tilde{D} \).

\( \tilde{G}(s) \) has the same number of inputs and outputs but a smaller state-space dimension: Low-order rational approximation to \( G(s) \).

\( \tilde{Y}(s) - Y(s) = \left( \tilde{G}(s) - G(s) \right) U(s) \)
Model Reduction by Tangential Interpolation

- Pick interpolation points \( \{\sigma_i\}_{i=1}^r \in \mathbb{C} \) together with the left directions \( \{c_i\}_{i=1}^r \in \mathbb{C}^p \) and the right directions \( \{b_i\}_{i=1}^r \in \mathbb{C}^m \):

\[
c_i^T \mathbf{G}(\sigma_j) = c_i^T \tilde{\mathbf{G}}(\sigma_j), \quad \mathbf{G}(\sigma_j)b_j = \tilde{\mathbf{G}}(\sigma_j)b_j, \quad (1)
\]

and

\[
c_i^T \mathbf{G}'(\sigma_j)b_j = c_i^T \tilde{\mathbf{G}}'(\sigma_j)b_j. \quad (2)
\]

- Construct

\[
\mathbf{V} = \begin{bmatrix} (\sigma_1 \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} b_1, & \cdots, & (\sigma_r \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} b_r \end{bmatrix} \in \mathbb{C}^{n \times r} \text{ and }
\]

\[
\mathbf{W} = \begin{bmatrix} (\sigma_1 \mathbf{E} - \mathbf{A}^T)^{-1} \mathbf{C}^T c_1 & \cdots & (\sigma_r \mathbf{E} - \mathbf{A}^T)^{-1} \mathbf{C}^T c_r \end{bmatrix} \in \mathbb{C}^{n \times r}
\]

- Then the interpolation conditions (1) and (2) are satisfied.

[Skelton et. al., 87], [Grimme, 97], [Gallivan et. al., 05]

- Interpolatory reduction of port-Hamiltonian systems:

[G./Polyuga/Beattie/vanderSchaft, 12], [Beattie/G.,11] and [Chaturantabut/Beattie/G.,13]
Recall in our case

\[
E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{21}^T \\ A_{21} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad C = [C_1 \ 0] \]

\( E_{11} \in \mathbb{R}^{n_1 \times n_1} \) and \( A_{21} \in \mathbb{R}^{n_2 \times n_1} \) have full rank and \( A_{21} E_{11}^{-1} A_{21}^T \) is nonsingular \( \iff \) Leading to an index-2 DAE.

Let \( G(s) \) be additively decomposed as: \( G(s) = G_{sp}(s) + P(s), \)

We will require that \( \tilde{G}(s) = \tilde{G}_{sp}(s) + \tilde{P}(s), \) with \( \tilde{P}(s) = P(s), \)

This will guarantee: \( G_{err}(s) = G(s) - \tilde{G}(s) = G_{sp}(s) - \tilde{G}_{sp}(s). \)

[Stykel,2004], [Mehrman/Stykel,2005], [Benner/Sokolov,2005], [Ali et al., 2013] [Heinkenschloss et al., 08]
\( \mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{P}(s). \)

We want \( \mathbf{G}(s) = \tilde{\mathbf{G}}_{sp}(s) + \tilde{\mathbf{P}}(s) \) with \( \tilde{\mathbf{P}}(s) = \mathbf{P}(s) \),

Problem reduces to: \( \tilde{\mathbf{G}}_{sp}(s) \) interpolates \( \mathbf{G}_{sp}(s) \).

\( \mathbf{P}_r \): the spectral projector onto the right deflating subspace of \( (\lambda \mathbf{E} - \mathbf{A}) \) corresponding to the finite eigenvalues.

\( \mathbf{P}_l \): Defined similarly for the left deflating subspace.

\( \mathbf{W}_\infty \) and \( \mathbf{V}_\infty \): Span, respectively, the right and left deflating subspaces of \( (\lambda \mathbf{E} - \mathbf{A}) \) corresponding to the infinite eigenvalues.
Theorem ([G./Stykel/Wyatt,12])

Given are $\mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$, interpolation points $\sigma \in \mathbb{C}$ and $\mu \in \mathbb{C}$; and the tangential directions $\mathbf{b} \in \mathbb{C}^m$ and $\mathbf{c} \in \mathbb{C}^p$. Define $\mathbf{V}_f$ and $\mathbf{W}_f$ such that

$$\mathbf{V}_f = \left[ (\sigma_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{P}_l \mathbf{b}_1, \cdots, (\sigma_r \mathbf{E} - \mathbf{A})^{-1}\mathbf{P}_l \mathbf{b}_r \right] \in \mathbb{C}^{n \times r} \text{ and}$$

$$\mathbf{W}_f = \left[ (\sigma_1 \mathbf{E} - \mathbf{A}^T)^{-1}\mathbf{P}_r^T \mathbf{C}^T \mathbf{c}_1 \cdots (\sigma_r \mathbf{E} - \mathbf{A}^T)^{-1}\mathbf{P}_r^T \mathbf{C}^T \mathbf{c}_r \right] \in \mathbb{C}^{n \times r}$$

Define $\mathbf{W} = [\mathbf{W}_f, \mathbf{W}_\infty]$ and $\mathbf{V} = [\mathbf{V}_f, \mathbf{V}_\infty]$, and construct $\tilde{\mathbf{G}}(s)$. Then,

1. $\mathbf{P}_r(s) = \mathbf{P}(s)$, and
2. $\mathbf{c}_i^T \mathbf{G}(\sigma_j) = \mathbf{c}_i^T \tilde{\mathbf{G}}(\sigma_j)$
   $$\mathbf{G}(\sigma_j) \mathbf{b}_j = \tilde{\mathbf{G}}(\sigma_j) \mathbf{b}_j \text{ for } j = 1, 2, \ldots, r.$$
   $$\mathbf{c}_i^T \mathbf{G}'(\sigma_j) \mathbf{b}_j = \mathbf{c}_i^T \tilde{\mathbf{G}}'(\sigma_j) \mathbf{b}_j$$
A Circuit Model

\[
E = \begin{bmatrix}
C_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C_2 & 0 & 0 & 0 \\
0 & 0 & 0 & L_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad
A = \begin{bmatrix}
-G_1 & G_1 & 0 & 0 & -1 \\
G_1 & -G_1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & -1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
B^T = \begin{bmatrix}
0 & 0 & 0 & 0 & -1 \\
\end{bmatrix} = C, \quad D = 0,
\]

\[
G(s) = C(sE - A)^{-1}B = \frac{sC_2 G_1}{s^2 C_2 L_1 G_1 + sC_2 + G_1} + \frac{SC_1}{P(s)}
\]

Borggaard and Gugercin

Interpolatory Model Reduction for Flow Control
Consider the model of an RLC circuit with \( n = 765 \) and index-2.

Reduce the order to \( r = 20 \) using complex interpolation points without the deflating subspaces:

<table>
<thead>
<tr>
<th>( \sigma_i )</th>
<th>( \mathbf{G}(\sigma_i) )</th>
<th>( \mathbf{G}_r(\sigma_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_1 )</td>
<td>( 9.8479 \times 10^{-3} + i3.4595 \times 10^{-3} )</td>
<td>( 9.8479 \times 10^{-3} + i3.4595 \times 10^{-3} )</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
<td>( 1.1586 \times 10^{-2} + i6.6549 \times 10^{-3} )</td>
<td>( 1.1586 \times 10^{-2} + i6.6549 \times 10^{-3} )</td>
</tr>
<tr>
<td>( \sigma_3 )</td>
<td>( 1.6518 \times 10^{-2} + i7.9917 \times 10^{-3} )</td>
<td>( 1.6518 \times 10^{-2} + i7.9917 \times 10^{-3} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \sigma_i )</th>
<th>( \mathbf{G}'(\sigma_i) )</th>
<th>( \mathbf{G}'_r(\sigma_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_1 )</td>
<td>( -1.1553 \times 10^{-12} - i3.7091 \times 10^{-14} )</td>
<td>( -1.1553 \times 10^{-12} - i3.7091 \times 10^{-14} )</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
<td>( -1.1045 \times 10^{-12} + i7.1250 \times 10^{-13} )</td>
<td>( -1.1045 \times 10^{-12} + i7.1250 \times 10^{-13} )</td>
</tr>
<tr>
<td>( \sigma_3 )</td>
<td>( -1.1846 \times 10^{-13} + i1.3335 \times 10^{-12} )</td>
<td>( -1.1846 \times 10^{-13} + i1.3335 \times 10^{-12} )</td>
</tr>
</tbody>
</table>
How do the Bode plots match?

- Polynomial part is completely missed.
Re-visit the previous example and apply the projection with deflating subspaces.

Requires computing $P_I$ and $P_r$. How to avoid this?
Define $\Pi = I - A_{12} \left( A_{21} E_{11}^{-1} A_{12} \right)^{-1} A_{21} E_{11}^{-1}$

$\Pi^2 = \Pi$, $\Pi E_{11} = E_{11} \Pi^T$, $\text{Null}(\Pi) = \text{Range}(A_{12})$.

Can be equivalently to reduced to ([Heinkenschloss et al., 08])

$$\Pi E_{11} \Pi^T \dot{v}_1(t) = \Pi A_{11} \Pi^T v_1(t) + \Pi B u(t)$$

$$y(t) = C v_1(t) + D_1 u(t) + D_2 \dot{u}(t)$$

We need $(\sigma_i \Pi E_{11} \Pi^T - \Pi A_{11} \Pi^T)^{-1} \Pi B b_i$

Define $\Pi E_{11} \Pi^T = \mathcal{E}$, $\Pi A_{11} \Pi^T = \mathcal{A}$, and, $\mathcal{B} = \Pi B$

Inverse defined on a restricted subspace:

$$(\sigma \mathcal{E} - \mathcal{A})^l (\sigma \mathcal{E} - \mathcal{A}) = \Pi^T, \quad (\sigma \mathcal{E} - \mathcal{A})(\sigma \mathcal{E} - \mathcal{A})^l = \Pi.$$

The vector $v_i = (\sigma \mathcal{E} - \mathcal{A})^l \mathcal{B} b_i$ solves

$$\begin{bmatrix}
\sigma \mathcal{E}_{11} + \mathcal{A}_{11} & \mathcal{A}_{12} \\
\mathcal{A}_{12}^T & 0
\end{bmatrix}
\begin{bmatrix}
v_i \\
z_i
\end{bmatrix}
= \begin{bmatrix}
\mathcal{B} b_i \\
0
\end{bmatrix}$$
Theorem (G./Stykel/Wyatt, 2013)

Given \( \{\sigma_i\} \in \mathbb{C}, \{b_i\} \in \mathbb{C}^m \) and \( \{c_i\} \in \mathbb{C}^p \), let \( v_i \) and \( w_i \) solve

\[
\begin{bmatrix}
\sigma_i \mathbf{E}_{11} - \mathbf{A}_{11} & \mathbf{A}_{21}^T \\
\mathbf{A}_{21} & 0
\end{bmatrix}
\begin{bmatrix}
v_i \\
z
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{B}_1 b_i \\
0
\end{bmatrix},
\]

\[
\begin{bmatrix}
\sigma_i \mathbf{E}_{11}^T - \mathbf{A}_{11}^T & \mathbf{A}_{21}^T \\
\mathbf{A}_{21} & 0
\end{bmatrix}
\begin{bmatrix}
w_i \\
q
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{C}^T c_i \\
0
\end{bmatrix}.
\]

for \( i = 1, \ldots, r \). Construct

\[
\mathbf{V} = [v_1, \ldots, v_r], \quad \text{and} \quad \mathbf{W} = [w_1, \ldots, w_r].
\]

Then \( \tilde{G}(s) = \mathbf{C} \mathbf{V} (s \mathbf{W}^T \mathbf{E}_{11} \mathbf{V} - \mathbf{W}^T \mathbf{A}_{11} \mathbf{V})^{-1} \mathbf{W}^T \mathbf{B}_1 + \mathbf{D}_1 \mathbf{u}(t) + \mathbf{D}_2 \dot{\mathbf{u}}(t) \) satisfies the required interpolation conditions and matches the polynomial part.
Interpolation points for $\mathcal{H}_2$ optimal approximation

\[
\|G\|_{\mathcal{H}_2} = \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|G(\omega)\|_F^2 \, d\omega \right)^{\frac{1}{2}}
\]

**Problem**

*Given $G(s)$, find $\tilde{G}(s)$ of order $r$ which solves:*

\[
\min_{\text{degree}(G_r)=r} \|G - G_r\|_{\mathcal{H}_2}
\]

**Solution**

- $\|G\|_{\mathcal{H}_2} = \sup_{u \neq 0} \frac{\|y\|_{\infty}}{\|u\|_2}$ for MISO and SIMO systems
- In general, $\|y - y_r\|_{\infty} \leq \|G - \tilde{G}\|_{\mathcal{H}_2} \|u\|_2$.
- Solution for the ODE case: [Meier/Luenberger, 67], [G./Antoulas/Beattie, 08] → Iterative Rational Krylov Algorithm: [G./Antoulas/Beattie, 08]
- Solution for the DAE case: [G./Stykel/Wyatt, 13]
**Theorem ([G./Stykel/Wyatt,13])**

For \( G(s) = G_{sp}(s) + P(s) \), let \( \tilde{G}(s) = \tilde{G}_{sp}(s) + \tilde{P}(s) \) minimize the \( \mathcal{H}_2 \) error \( \|G - \tilde{G}\|_{\mathcal{H}_2} \). Then, \( \tilde{P}(s) = P(s) \), and, hence \( \tilde{G}_{sp}(s) \) minimizes the \( \mathcal{H}_2 \) error \( \|G_{sp} - \tilde{G}_{sp}\|_{\mathcal{H}_2} \). Moreover, let \( \tilde{G}_{sp}(s) = C_{sp}(sE_{sp} - \tilde{A}_{sp})^{-1}\tilde{B}_{sp} \).

Suppose that the reduced-order pencil \( \lambda \tilde{E}_{sp} - \tilde{A}_{sp} \) has distinct eigenvalues \( \{\tilde{\lambda}_i\}_{i=1}^r \), i.e., \( \tilde{G}_{sp}(s) = \sum_{i=1}^r \frac{1}{s-\tilde{\lambda}_i} \tilde{c}_i \tilde{b}_i^T \). Then, for \( i = 1, \ldots, r \),

\[
G(-\tilde{\lambda}_i)\tilde{b}_i = \tilde{G}(-\tilde{\lambda}_i)\tilde{b}_i, \quad \tilde{c}_i^T G(-\tilde{\lambda}_i) = \tilde{c}_i^T \tilde{G}(-\tilde{\lambda}_i),
\]

and \( \tilde{c}_i^T G'(-\tilde{\lambda}_i)\tilde{b}_i = \tilde{c}_i^T \tilde{G}'(-\tilde{\lambda}_i)\tilde{b}_i \).

- Iterate on the interpolation points and directions until convergence.
Iterative Rational Krylov Algorithm (IRKA): Algorithm (G./Antoulas/Beattie [2008])

1. Choose \{\sigma_1, \ldots, \sigma_r\}, \{\hat{b}_1, \ldots, \hat{b}_r\} and \{\hat{c}_1, \ldots, \hat{c}_r\}

2. \[
V = \begin{bmatrix}
(\sigma_1 E - A)^{-1}B\hat{b}_1 & \cdots & (\sigma_r E - A)^{-1}B\hat{b}_r
\end{bmatrix}
\]
   \[
W = \begin{bmatrix}
(\sigma_1 E - A^T)^{-1}C^T\hat{c}_1 & \cdots & (\sigma_r E - A^T)^{-1}C^T\hat{c}_r
\end{bmatrix}
\]

3. while (not converged)

   1. \( \tilde{A} = W^T AV, \tilde{E} = W^T EV, B = W^T B, \ and \ C = CV \)

   2. Compute \( \tilde{G}(s) = \sum_{i=1}^{r} \frac{c_i b_i^T}{s - \lambda_i} \)

   3. \( \sigma_i \leftarrow -\lambda_i, \hat{b}_i \leftarrow b_i \ and \ \hat{c}_i \leftarrow c_i \)

   4. \[
V = \begin{bmatrix}
(\sigma_1 E - A)^{-1}B\hat{b}_1 & \cdots & (\sigma_r E - A)^{-1}B\hat{b}_r
\end{bmatrix}
\]
   \[
W = \begin{bmatrix}
(\sigma_1 E - A^T)^{-1}C^T\hat{c}_1 & \cdots & (\sigma_r E - A^T)^{-1}C^T\hat{c}_r
\end{bmatrix}
\]

4. \( \tilde{A} = W^T_r AV_r, \tilde{E} = W^T_r EV_r, \tilde{B} = W^T_r B, \tilde{C} = CV_r, \tilde{D} = D. \)
Discretization of Navier-Stokes/Oseen equations

Figure: Discretization by Taylor-Hood Finite Elements

- Leads to 21,390 velocity degrees of freedom ($x_1$),
- and 2,777 pressure degrees of freedom ($x_2$).
- Solved at $Re_d = 60$ ($\mu = 1/Re$)
Recall $n_1 = 21390$ and $n_2 = 2777$
We reduce the order to $r = 60$ using interpolatory projection.

Relative $\mathcal{L}_\infty$ error $= 1.5406 \times 10^{-5}$
For this $Re_d$, the full-model has two unstable poles. These unstable poles are captured very accurately.

$$
\lambda_{\text{unstable}}(G(s)) : 5.248019596820730 \times 10^{-2} \pm i 7.672028760928972 \times 10^{-1}
$$

$$
\lambda_{\text{unstable}}(\tilde{G}(s)) : 5.248030491505502 \times 10^{-2} \pm i 7.672029050490372 \times 10^{-1}
$$

Solve the reduced LQR problem and compute the functional gains:

**Figure**: Horizontal (left) and Vertical (right) Components
Open Loop Simulation
Closed Loop: From $t = 20$
Two-cylinder case

Figure: Discretization by Taylor-Hood Finite Elements

- Leads to 132476 velocity degrees of freedom ($x_1$),
- and 16691 pressure degrees of freedom ($x_2$).
- Solved at $Re_d = 60$ ($\mu = 1/Re$)
Model Reduction for the Two-cylinder Case

- Recall $n_1 = 132476$ and $n_2 = 16691$, $n = n_1 + n_2 = 149167$.
- We reduce the order to $r = 150$ using interpolatory projection.

Relative $\mathcal{L}_\infty$ error $= 6.3980 \times 10^{-6}$
Unstable poles are, once again, captured very accurately.

\[
\lambda_{\text{unstable}}(G(s)) : 3.973912561638801 \times 10^{-2} \pm i 7.498560362688469 \times 10^{-1}
\]

\[
\lambda_{\text{unstable}}(\tilde{G}(s)) : 3.973912526082657 \times 10^{-2} \pm i 7.498560367601876 \times 10^{-1}
\]

Solve the reduced LQR problem and compute the functional gains:

Figure: Horizontal (left) and Vertical (right) Components
Closed Loop: Controlled from $t = 100$
Control Inputs

control inputs

top cylinder
bottom cylinder

tangential velocity
time
Figure: Discretization by Taylor-Hood Finite Elements

- Leads to 299338 velocity degrees of freedom \( (x_1) \),
- and 37714 pressure degrees of freedom \( (x_2) \).
- Solved at \( Re_d = 100 \ (\mu = 1/Re) \)
Model Reduction for the Two-cylinder Case

- Recall $n_1 = 299338$ and $n_2 = 37714$, $n = n_1 + n_2 = 337052$.
- We reduce the order to $r = 170$ using interpolatory projection.

![Singular Value Plots of $G(s)$ and $\tilde{G}(s)$](image)

- Relative $\mathcal{L}_\infty$ error $= 1.5154 \times 10^{-5}$
Unstable Poles

- For $Re_d = 100$, the full-model has seven unstable poles.

- The unstable poles of the reduced model $\tilde{G}(s)$:
  
  $\begin{align*}
  1.245178576584041 \times 10^{-1} & \pm i \ 7.507209792650027 \times 10^{-1} \\
  3.195053261722973 \times 10^{-2} & \pm i \ 8.505319185007424 \times 10^{-1} \\
  8.325502142822423 \times 10^{-3} & \pm i \ 7.314950149341377 \times 10^{-1} \\
  2.580915637572443 \times 10^{-2} &
  \end{align*}$

- Accurate to 5 significant digits

- Unstable poles are, once again, captured accurately.

- We follow similarly and solve the reduced LQR problem.
Convergence of Gain with Model Size

Figure: Gain $h_1$ for $r = 120$

Figure: Gain $h_1$ for $r = 140$

Figure: Gain $h_1$ for $r = 170$
Open Loop Simulation: disturbance for $t \in (0, 2\pi)$
Closed loop from $t = 10$
Nonlinear dynamical systems

- \( \dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad y(t) = Cx(t) \)

- The most common and a rather effective approach: Proper Orthogonal Decomposition (POD)

- Pick your favorite input \( u(t) \), run the system from \( t = 0 \) to \( t_N = T \) and construct a snapshot matrix:

\[
X = [x(t_1), x(t_2), \ldots, x(t_N)] \in \mathbb{R}^{n \times N}
\]

- Compute the SVD of \( X \): \( X = U\Sigma Z^T \)

- Choose \( V \) as the leading \( r \) columns of \( U \).

- \( \dot{x}_r(t) = V^TAVx_r(t) + V^Tf(Vx_r(t), u(t)), \quad y_r(t) = CVx_r(t) \)
- Input-dependent reduced-order model.

- The reduced-model is usually only as good as the information in $X$.

- For linear dynamics, $u(t)$ did not enter into the model reduction step.

- Can we mimic the linear case for some special cases?

- How to extend the idea of transfer function to the nonlinear setting?
Bilinear Systems

\[ \zeta : \begin{cases} \dot{x}(t) = Ax(t) + Nx(t)u(t) + bu(t) \\ y(t) = c^T x(t) \end{cases}, \]

where \( A, N \in \mathbb{R}^{n \times n}, b, c \in \mathbb{R}^n, u(t) \in \mathbb{R} \) and \( x(t) \in \mathbb{R}^n \).

The output \( y(t) \) has the Volterra series representation

\[ y(t) = \sum_{k=1}^{\infty} \int_0^\infty \cdots \int_0^\infty h_k(t_1, \ldots, t_k) u(t - t_1 - t_2 - \cdots - t_k) \cdots u(t - t_k) \, dt_k \cdots dt_1, \]

where \( h_k(t_1, \ldots, t_k) = c^T e^{At_k} Ne^{At_{k-1}} N \cdots Ne^{A t_1} b. \)

\[ \mathcal{L}[h_k(t_1, \ldots, t_k)] = H_k(s_1, s_2, \ldots, s_k) \]
\[ = c^T (s_k I - A)^{-1} N (s_{k-1} I - A)^{-1} N \cdots N (s_1 I - A)^{-1} b. \]
Model Reduction in the Petrov-Galerkin Framework

- Given
  \[\begin{aligned}
  \dot{x}(t) &= Ax(t) + Nx(t)u(t) + bu(t) \\
y(t) &= c^T x(t)
  \end{aligned}\]
  of dimension \(n\).

- For \(r \ll n\), find
  \[\begin{aligned}
  \dot{x}_r(t) &= \tilde{A}x_r(t) + \tilde{N}x_r(t)u(t) + \tilde{b}u(t) \\
y_r(t) &= \tilde{c}^T x_r(t)
  \end{aligned}\]
such that \(y_r(t) \approx y(t)\).

- Define \(\zeta_r\) via the projected equations:
  \[\begin{aligned}
  \dot{x}_r(t) &= W^T AVx_r(t) + W^T NVx_r(t)u(t) + W^T bu(t) \\
y_r(t) &= c^T V x_r(t)
  \end{aligned}\]
What to interpolate

- Construct $\mathbf{V}$ and $\mathbf{W}$ so that

$$H_k(\sigma_1, \sigma_1, 2, \ldots, \sigma_1, \ldots, k) = \tilde{H}_k(\sigma_1, \sigma_1, 2, \ldots, \sigma_1, \ldots, k)$$

for $k = 1, \ldots, N$. [Phillips, 2002], [Bai and Skoogh, 2006], [Breiten and Damm, 2009].

⇒ The leading $N$ subsystems of $\tilde{H}(s)$ interpolates those of $H(s)$.

- Optimal $\mathcal{H}_2$ reduction for bilinear systems: [Benner/Breiten, 11]: B-IRKA
  - Input-independent optimal model reduction for a nonlinear system.
  - Significantly more accurate approximations than the subsystem interpolation methods and better performance than bilinear balanced truncation.

- Interpolate the infinite-Volterra series, not just the subsystems:
  [Flagg/G., 15]:
  - Solve bilinear Sylvester equations
  - B-IRKA interpolates the infinite-Volterra series.
Consider the 1D Burgers equation over $[0, 1] \times [0, t_f]$.

\[
\begin{align*}
v_t(x, t) + v(x, t) \cdot v_x(x, t) &= \nu \cdot v_{xx}(x, t), \\
v(0, t) &= u(t), \quad v_x(1, t) = 0, \quad v(x, 0) = v_0(x) = 0
\end{align*}
\]

A finite difference discretization yields

\[
\dot{x}(t) = Ax(t) + H(x(t) \otimes x(t)) + Nx(t)u(t) + bu(t)
\]

\[
y(t) = c^T x(t)
\]

where

\[
A, N \in \mathbb{R}^{n \times n}, \quad H \in \mathbb{R}^{n \times n^2}, \quad b, c \in \mathbb{R}^n
\]

In our tests, we took $\nu = 0.02$ and $n = 1500$.\[\]
Reduced Order Model (ROM)

- Construct

\[
\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{H}(\tilde{x}(t) \otimes \tilde{x}(t)) + \tilde{N}\tilde{x}(t)u(t) + \tilde{b}u(t)
\]

\[
\tilde{y}(t) = \tilde{c}^T\tilde{x}(t)
\]

via projection

\[
\tilde{A} = V^TAV, \quad \tilde{H} = V^TH(V \otimes V), \quad \tilde{N} = V^TNV
\]

\[
\tilde{b} = V^Tb, \quad \tilde{c} = V^Tc
\]

- Subsystem interpolation: [Gu,11], [Benner/Breiten,15]

- Here, we will use optimal interpolation subspaces from the linearized model.

We test the technique against several input functions and various values of $r_W$ and $r_V$.

- First, we generate ROMs using POD and one-sided IRKA.
- Next we picked $r_W = 7, \ldots, 10$.
- For each $r_W$, we calculated a ROM for each of $r_V = (r - r_W), \ldots, (r - 1)$.
- For each ROM, the output error was calculated.
Error plots for \( u_1(t) = \cos(\pi t) \)
Output plot for $u_1(t)$ with $r_w = 7$
Error plots for $u_2(t) = 2 \sin(\pi t)$

IRKA V+W Error Comparison for Control $u_2(t)$

Relative Error

# IRKA V Vectors

IRKA V+W Error Comparison for Control $u_2(t)$

$r_w = 7$
$r_w = 8$
$r_w = 9$
$r_w = 10$

POD error
IRKA V only
QBMOR V only
Output plots for $u_2(t) = 2 \sin(\pi t)$ with $r_W = 7$
Plot of control function $u_3(t)$

Step Function

Control $u_3$

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Interpolatory Model Reduction for Flow Control
Error plots for $u_3(t)$

IRKA V+W Error Comparison for Control $u_3(t)$

- $r_w = 7$
- $r_w = 8$
- $r_w = 9$
- $r_w = 10$

POD error
IRKA V only
QBMOR V only
Output plots for $u_3(t)$

IRKA V+W Galerkin ($r=15$, $r_v=8$, $r_w=7$)

- Actual
- IRKA V+W (1S)
- POD

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Interpolatory Model Reduction for Flow Control
Conclusions and Future Work

- Interpolatory model reduction for DAEs combined with LQR design for flow control
- Computationally efficient framework
- Unstable poles captured accurately
- Incorporating optimal linear subspaces into reducing nonlinear models
- Establish the connection to rational Krylov methods for eigenvalue problems.
- Test the performance for higher Reynolds numbers.
- Choice of $C$
Index-2 example: Oseen equations

- Data from [Heinkenschloss et al., 08]

- Discretized the Oseen equations: describing the flow of a viscous and incompressible fluid in a domain $\Omega \in \mathbb{R}^2$ representing a channel with a backward facing step.

$$\begin{align*}
E_{11}, A_{11} &\in \mathbb{R}^{5520 \times 5520}, \\
A_{12}, A_{21}^T &\in \mathbb{R}^{5520 \times 761}, \\
B_1 &\in \mathbb{R}^{5520 \times 6}, \\
B_2 &\in \mathbb{R}^{761 \times 6}, \\
C_1 &\in \mathbb{R}^{2 \times 5520}, \\
C_2 &\in \mathbb{R}^{2 \times 761}, \text{ and } D = 0.
\end{align*}$$

- Reduced to order $r = 20$ using interpolatory $\mathcal{H}_2$ method for index-2 DAEs.

- Also compared with balanced truncation
Relative $\mathcal{H}_\infty$-error: \[ \frac{\|G_{sp} - \tilde{G}_{sp}\|_{\mathcal{H}_\infty}}{\|G_{sp}\|_{\mathcal{H}_\infty}}. \]

IRKA – DAE : $8.9663 \times 10^{-6}$  
BT : $3.3284 \times 10^{-6}$
Compare the full and reduced model for the input selections $u_i(t) = \sin(6it)$ for $i = 1, \ldots, 6$.

**Figure:** Oseen equation: (left) time domain response for $u_i(t) = \sin(6it)$; (right) error in time domain response for $u_i(t) = \sin(6it)$. 