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LQG-Balanced Truncation Low-Order Controller for Stabilization of Laminar Flows

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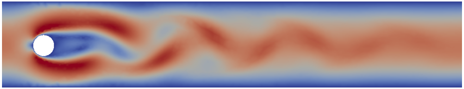
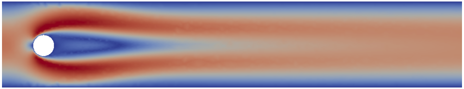
Outline

Problem Statement



- Cylinder wake at moderate *Reynolds* numbers
- The steady state is a solution, but unstable
- Goal: Stabilizing feedback controller that works in experiments
- Thus, the simulation needs to cope with:
  - ➙ limited measurements
  - ➙ short evaluation times
  - ➙ external perturbations
  - ➙ actuation at the boundary



Model Based and Reduced Controller

We propose a controller, that is a simultaneous application of

- a linearization about the steady state
  - ➙ to directly attack the deviations
- a *Kalman filter*
  - ➙ estimate the state using a few measurements
- an *LQG regulator*
  - ➙ stabilize the linearized system
- and *Balanced Truncation*
  - ➙ reduce the linearized and stable system

Expectations and Limitations

The proposed controller is based on a linearized model

➙ we expect a good performance for small deviations

and is designed to work for

- ✓ limited state information
- ✓ fast and unstable dynamics
- ✓ high dimensionality
- ✓ boundary control.

Related Work

Jonckheere and Silverman. A new set of invariants for linear systems – application to reduced order compensator design.
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Illustration Example

Stabilization with a Regulator

Consider the minimal but unstable linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx. \end{aligned}$$

The positive definite solution X_c to the control Riccati equation

$$A^T X_c + X_c A - X_c B B^T X_c + C C^T = 0$$

defines a stabilizing feedback, i.e.

$$\dot{x} = (A - BB^T X_c)x,$$

is asymptotically stable.

Illustration Example

Balanced Truncation of the Stabilized System

For stable linear time-invariant systems like,

$$\dot{x} = (A - BB^T X_c)x, \quad y = Cx,$$

Balanced Truncation is the first candidate for model reduction.

1. Compute the *controllability* and the *observability* Gramians G_c and G_o, e.g. via Lyapunov equations

$$(A - BB^T X_c)^T G_c + G_c(A - BB^T X_c) + C^T C = 0$$

2. From G_c and G_o one can derive a state transformation such that the transformed Gramians fulfill

$$G_c = G_o = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$$

3. Truncate all states associated with $\sigma_i < \sigma_{\texttt{tol}}$.

Illustration Example

Important Observations

- For some parameters: $G_c = X_c$ and $G_o = X_o$
  - ➙ Stabilization and truncation in one step
- There is an a-priori error bound for the truncation
  - ➙ $\|H - H_{\mathtt{tol}}\|_{\mathcal{H}^\infty} \leq 2 \sum_{\sigma_i < \sigma_{\mathtt{tol}}} \sigma_i$
- For constrained systems (like the Navier-Stokes equations) similar procedures work
  - ➙ see below
- An observer can be reduced simultaneously
  - ➙ application for output feedback

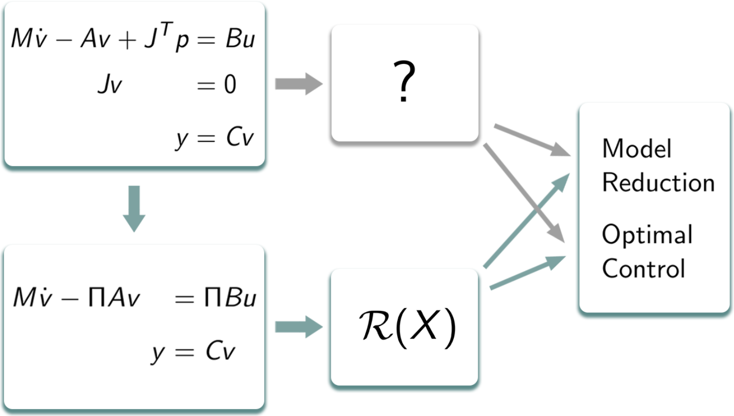
Computational Challenges

The major effort lies in the computation of X_o and X_c, because of

1. high-dimensionality: X_c, $X_o \in \mathbb{R}^{n_v,n_v}$
   $\rightarrow$ n_v is the dimension of the state $v(t)$
2. nonlinearity of the Riccati equation
   $\rightarrow$ a good initial guess for a Newton iteration is needed
3. differential algebraic structure of the state equations
   $\rightarrow$ X_c, X_o need to obey divergence constraints

Constrained Riccati Equations

LQGBT Based Controller



with A, M, $\Pi \in \mathbb{R}^{n_x, n_x}$, $J \in \mathbb{R}^{n_v, n_p}$, $B \in \mathbb{R}^{n_x, n_u}$, and $C \in \mathbb{R}^{n_y, n_x}$.

Constrained Riccati Equations

For Flow Equations

$$
\begin{aligned}
M\dot{v} - Av + J^T p &= Bu \\
Jv \qquad &= 0 \\
y &= Cv
\end{aligned}
$$

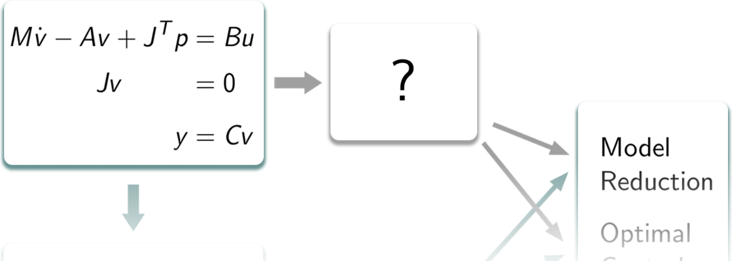
↓

$$
\begin{aligned}
M\dot{v} - \Pi A v \quad &= \Pi B u \\
y &= Cv
\end{aligned}
$$

- Projection onto the manifold of the constraints
- gives an ODE
- equivalent in theory
- but problematic in practice
  - numerically infeasible
  - systematic errors may be introduced
  - structure is not preserved

Constrained Riccati Equations

for constrained dynamics



Derivation of the constrained Riccati equations

- directly via optimality conditions,
  - → [Kunkel, Mehrmann '08], [Kurina, März '07], [Heiland '14]
- reformulation of the ODE related system,
  - → see below, [Benner, Heiland '14]
- or reformulation of the numerical schemes
  - → [Heinkenschloss, Sorensen, Sun '08], [Gugercin, Stykel, Wyatt '13].

Constrained Riccati Equations

Projected Riccati Equation

To define, e.g., the *Linear-Quadratic Regulator*, one needs a solution to the associated *control* Riccati equation of the form

$$\Pi A^T \Pi^T X M + M^T X \Pi A \Pi^T - M^T X \Pi B B^T \Pi^T X M + \Pi C^T C \Pi^T = 0$$

for $X \in \mathbb{R}^{n_v, n_v}$.

Equivalence to Projected Riccati Equations

Lemma

Let M be invertible, J have full rank, and $\Pi := I - J^T(JM^{-1}J^T)^{-1}JM^{-1}$. The matrix $X \in \mathbb{R}^{n_v,n_v}$ solves,

$$\Pi A^T \Pi^T XM + M^T X \Pi A \Pi^T - M^T X \Pi BB^T \Pi^T XM + \Pi C^T C \Pi^T = 0$$

if it solves

$$\begin{aligned} A^T XM + M^T XA - M^T XBB^T XM + \\ MYJ^T + JY^T M^T + CC^T &= 0, \\ JXM^T &= 0, \\ MXJ^T &= 0, \end{aligned}$$

for a suitable $Y \in \mathbb{R}^{n_v,n_p}$.

Low-Rank Approximations

How to obtain approximations to a solution of

$$
\begin{aligned}
A^T XM + M^T XA - M^T XBB^T XM + \\
MYJ^T + JY^T M^T + CC^T &= 0, \\
JXM^T &= 0.
\end{aligned}
$$

1. Factorize the solution $X = ZZ^H$,
2. apply a *low-rank Newton-ADI iteration* [Benner, Li, Penzl '08] to the constrained Riccati equation [Heiland '14], and
3. obtain skinny factors Z_{n_k}, that approximate $X \approx Z_{n_k} Z_{n_k}^H$.

Constrained Riccati Equations

Applications

Same idea and result for

- Lyapunov equations,
  - e.g. for Balanced Truncation,
- *Filter* Riccati equations,
  - e.g. for observer design or LQG-Balanced Truncation,
- and Differential Riccati equations,
  - e.g. for finite time-horizon control.

Numerical Example

We consider spatially discretized *Navier-Stokes* equations with boundary control u and observation $y = Cv$

$$\begin{aligned} M\dot{v} &= -N(v)v - \frac{1}{Re}Lv + J^T p - Bu + f, \\ 0 &= Jv - g, \\ v(0) &= \alpha, \\ y &= Cv, \end{aligned}$$

where

- α is the steady-state solution and
- the input operator B models Dirichlet conditions via approximating Robin conditions

Definition of the Input Operator

- Control through injection and suction at outlets Γ_{c_1}, Γ_{c_2} located at the cylinder periphery at $\pm\pi/3$.

- Prescribe Dirichlet conditions for the velocity

$$v = g_1(x)u_1(t), \quad v = g_2(x)u_2(t)$$

at Γ_{c_1} and Γ_{c_2}, where $g_{1/2}$ are the shape functions and $u_{1/2}$ are the magnitudes of the controls.

  - Use a small γ to relax the Dirichlet conditions to Robin conditions at $\Gamma_{1/2}$:

$$v \approx g_{1/2}u_{1/2} + \gamma(\frac{1}{Re}\frac{\partial v}{\partial n} - pn)$$

  - that are *naturally* included in Finite Element discretizations.
  - For other approaches see [Benner, Heiland '15].

Defining the Controller

1. Consider the linearization about α

$$
\begin{aligned}
M\dot{v} &= A_\alpha v + J^T p - Bu + f, \quad v(0) = \alpha, \\
0 &= Jv, \\
y &= Cv.
\end{aligned}
$$

2. Compute X_c and X_o which solve the associated *control* and *filter Riccati equations* to define the state estimate $\hat{x}$ and the regulator u as

$$
\begin{aligned}
M\dot{\hat{x}} &= \hat{A}_\alpha \hat{x} + X_o M C^T (y - C\alpha), \\
u &= -B^T M X_c \hat{x},
\end{aligned}
$$

   with $\hat{x}(0) = 0$ and $\hat{A}_\alpha$ denoting the observer dynamics.

3. Balance and truncate X_o and X_c to define a reduced observer

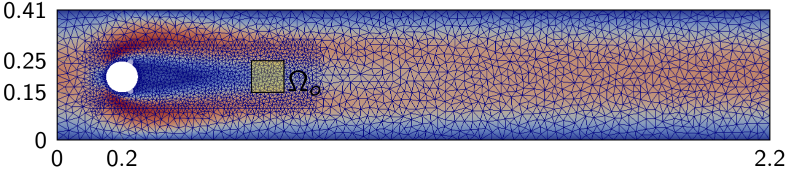
Reduced Closed Loop System

After the truncation, we arrive at

$$\begin{aligned} M\dot{v} &= -N(v)v - \frac{1}{Re}Lv + J^T p - BB_k^T X_{ck}\hat{x}_k + f, \\ 0 &= Jv - g, \\ v(0) &= \alpha, \\ y &= Cv, \\ \dot{\hat{x}}_k &= (A_{\alpha k} - X_{ok}C_k^T C_k - B_k B_k^T X_{ck})\hat{x}_k + X_{ok}C_k^T(y - y_\alpha), \\ \hat{x}_k(0) &= 0, \end{aligned}$$

where, in particular, $A_{\alpha k}$, B_k, C_k, X_{ck}, X_{ok} define the reduced system for $\hat{x}_k(t) \in \mathbb{R}^{n_k}$ with $n_k \ll n_v$ (dimension of $v(t)$).

Simulation Setup



- 2D cylinder wake
- Navier-Stokes Equations
- $Re = 100$
- *Taylor-Hood* finite elements
- 30000 velocity nodes
- Boundary control at 2 outlets
- distributed observation with 6 degrees of freedom
- LQGBT-reduced order observer and controller of state dimension $n_k = 13$
- Target: stabilization of the steady-state solution

Simulation Results

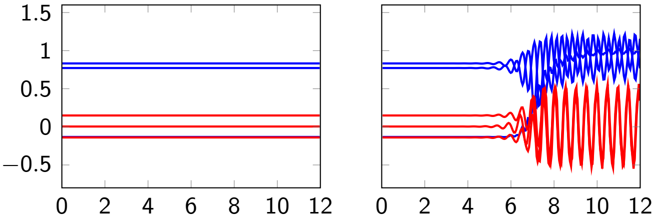
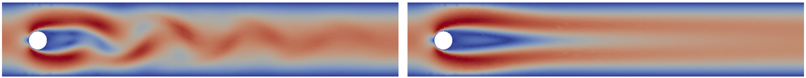


Figure : Measured signal y versus time $t \in [0, 12]$ of the perturbed closed loop system with a reduced controller of dimension $n_k = 13$ (left), compared to the response of the uncontrolled system (right). Blue corresponds to the x-component of the velocity and red to y-component. Below, a snapshot of the magnitude of the velocity solutions at $t = 12$.



Summary and Conclusion

- The general LQGBT approach has been applied to controller design Navier-Stokes equations
- The DAE structure is accounted for using constraint Riccati equations
- The resulting controller is of very small dimension and works for limited state information
- The numerical approximation of the controller requires advanced methods for solving large-scale Riccati equations
- Successful application in boundary control of the cylinder wake

Thank you for your attention!

More Literature

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