Well-posedness and stabilization of energy-preserving partial differential equations

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Standard Hamiltonian equations for a mechanical system

Hamiltonian formulation was introduced in 1833 by William Rowan Hamilton.

Hamiltonian system

$$\dot{q} = +\frac{\partial H}{\partial p}(q,p)$$
$$\dot{p} = -\frac{\partial H}{\partial q}(q,p)$$

$H(p,q)$	$=$	Hamiltonian, total energy of the system
p	$=$	vector of generalized momenta
q	$=$	generalized configuration coordinates

Hamiltonian system $$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial H}{\partial (q,p)}(q,p)$$

Port-Hamiltonian systems

Port -Hamiltonian system (Maschke & van der Schaft ’95)

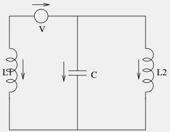
$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)f \\ e &= g^T(x)\frac{\partial H}{\partial x}(x) \end{aligned}$$

$J(x) = -J^T(x)$, $x \in X \subset \mathbb{R}^n$, X state-space manifold

Examples

- Constrained Hamiltonian equations: Kinematic constrains like $A^T(q)\dot{q} = 0$.

- Network models like



Infinite-dimensional Port-Hamiltonian systems

Formally

$$\dot{x} \quad = \quad J(x)\frac{\partial H}{\partial x}(x)$$

J is a formally skew-adjoint operator on a function space.

What is $\frac{\partial H}{\partial x}$ in infinite dimensions?

Finite-dimensional systems: $\frac{\partial H}{\partial x} \triangleq$ gradient

Infinite-dimensional systems: $\frac{\partial H}{\partial x} \triangleq \frac{\delta H}{\delta x} \triangleq$ variational derivative

Example: Quadratic Hamiltonian

$$H(x) = \frac{1}{2}\int x(\zeta)^T \mathcal{H}(\zeta) x(\zeta)\, d\zeta$$

Then we have: $\frac{\delta H}{\delta x}(x) = \mathcal{H}x.$

Infinite-dimensional port-Hamiltonian systems

Literatur

- Port-Hamiltonian formulation of distributed-parameter systems
  (van der Schaft & Maschke, Scherpen & Voss, ...)

In this talk: Analysis of Infinite-dimensional port-Hamiltonian systems

Class of Port-Hamiltonian systems

$$\frac{\partial x}{\partial t}(\zeta,t) = \underbrace{\left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)}_{J(x)} \underbrace{[\mathcal{H}(\zeta)x(\zeta,t)]}_{\frac{\delta H}{\delta x}}$$

$$H(x(\cdot,t)) = \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta) x(\zeta,t) d\zeta.$$

- P_1 is an invertible, symmetric real $n \times n$-matrix,
- P_0 is an skew-symmetric real $n \times n$-matrix,
- $\mathcal{H}(\zeta)$ is a symmetric, invertible $n \times n$-matrix with $mI \leq \mathcal{H}(\zeta) \leq MI$ for some $m, M > 0$.

The wave equation

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right]$$

Energy

$$\begin{aligned} H(t) &= \frac{1}{2}\int_a^b \rho(\zeta)\left(\frac{\partial w}{\partial t}(\zeta,t)\right)^2 + T(\zeta)\left(\frac{\partial w}{\partial \zeta}(\zeta,t)\right)^2 d\zeta \\ &= \frac{1}{2}\int_a^b \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix}^T \begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{bmatrix} \begin{bmatrix} x_1(\zeta,t) \\ x_2(\zeta,t)\end{bmatrix} d\zeta. \end{aligned}$$

$x_1 := \rho\frac{\partial w}{\partial t}$ (the momentum), $x_2 := \frac{\partial w}{\partial \zeta}$ (the strain)

$$\frac{\partial}{\partial t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(\zeta,t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{=P_1} \frac{\partial}{\partial \zeta}\left[\underbrace{\begin{bmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta)\end{bmatrix}}_{=\mathcal{H}} x(\zeta,t)\right]$$

The Timoshenko beam

$$
\begin{aligned}
\rho(\zeta)\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left[K(\zeta)\left[\frac{\partial w}{\partial \zeta}(\zeta,t)-\phi(\zeta,t)\right]\right] \\
I_\rho(\zeta)\frac{\partial^2 \phi}{\partial t^2}(\zeta,t) &= \frac{\partial}{\partial \zeta}\left[EI(\zeta)\frac{\partial \phi}{\partial \zeta}\right]+K(\zeta)\left[\frac{\partial w}{\partial \zeta}(\zeta,t)-\phi(\zeta,t)\right],
\end{aligned}
$$

$w(\zeta,t)$ = is transverse displacement of the beam

$\phi(\zeta,t)$ = is rotation angle of a filament of the beam

We choose

$x_1(\zeta,t)$	=	$\frac{\partial w}{\partial \zeta}(\zeta,t)-\phi(\zeta,t)$	shear displacement
$x_2(\zeta,t)$	=	$\rho(\zeta)\frac{\partial w}{\partial t}(\zeta,t)$	momentum
$x_3(\zeta,t)$	=	$\frac{\partial \phi}{\partial \zeta}(\zeta,t)$	angular displacement
$x_4(\zeta,t)$	=	$I_\rho(\zeta)\frac{\partial \phi}{\partial t}(\zeta,t)$	angular momentum

The Timoshenko beam

Timoshenko beam

$$\frac{\partial x}{\partial t}(\zeta,t) = \left(P_1\frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}x(t)]$$

$$H(x(\cdot,t)) = \frac{1}{2}\int_a^b x(\zeta,t)^T\mathcal{H}(\zeta)x(\zeta,t)d\zeta.$$

with $P_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $P_0 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and

$$\mathcal{H}(\zeta) = \begin{bmatrix} K(\zeta) & 0 & 0 & 0 \\ 0 & \frac{1}{\rho(\zeta)} & 0 & 0 \\ 0 & 0 & EI(\zeta) & 0 \\ 0 & 0 & 0 & \frac{1}{I_\rho(\zeta)} \end{bmatrix}$$

Port-Hamiltonian partial differential equations

$$\frac{\partial x}{\partial t}(\zeta,t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}x(t)]$$
$$H(x(\cdot,t)) = \frac{1}{2}\int_a^b x(\zeta,t)^T \mathcal{H}(\zeta) x(\zeta,t)\,d\zeta.$$

Question: Which boundary conditions lead to unique solutions?

Example

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right]$$
$$\frac{\partial w}{\partial t}(0,t) = T(1)\frac{\partial w}{\partial \zeta}(1,t) = 0$$

Abstract Cauchy problem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \qquad t \geq 0.$$

Assumptions:

- X is a Hilbert space
- $A : D(A) \subset X \to X$ generates a C_0-semigroup $(e^{At})_{t\geq 0}$ on X, i.e.
  - For every $t \geq 0$: e^{At} is a linear bounded operator on X
  - $e^{A0} = I$, $e^{A(t+\tau)} = e^{At}e^{A\tau}$
  - $\|e^{At}x_0 - x_0\|$ converges to 0 for $t \to 0$
  - $Ax = \lim_{h\to 0+} \frac{1}{h}(e^{Ah}x - x)$ for $x \in D(A)$
  - $D(A) = \{x \in X \mid \lim_{h\to 0+} \frac{1}{h}(e^{Ah}x - x) \text{ exists}\}$

The mild solution is given by $x(t) = e^{At}x_0$.

If $x_0 \in D(A)$ then $x(\cdot)$ is the classical solution

Port-Hamiltonian partial differential equations

$$\begin{aligned} \frac{\partial x}{\partial t}(\zeta,t) &= \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}(\zeta)x(\zeta,t)] \\ W\begin{bmatrix}\mathcal{H}(b)x(b,t)\\ \mathcal{H}(a)x(a,t)\end{bmatrix} &= 0 \end{aligned}$$

Hilbert space: $X = L^2(a,b;\mathbb{R}^n)$

Inner product: $\langle x, y\rangle = \frac{1}{2}\int_a^b x(\zeta)^T \mathcal{H}(\zeta) y(\zeta)\, d\zeta$

$$Ax = \left(P_1 \frac{d}{d\zeta} + P_0\right)[\mathcal{H}x]$$

$$D(A) = \left\{ x \in X \mid \frac{d}{d\zeta}\mathcal{H}x \in X, W\begin{bmatrix}\mathcal{H}(b)x(b)\\ \mathcal{H}(a)x(a)\end{bmatrix} = 0 \right\}$$

Port-Hamiltonian partial differential equations

$$X = L^2(a,b;\mathbb{R}^n), \qquad \langle x, y\rangle = \frac{1}{2}\int_a^b x(\zeta)^T \mathcal{H}(\zeta) y(\zeta)\, d\zeta$$

$$Ax = \left(P_1 \frac{d}{d\zeta} + P_0\right)[\mathcal{H}x]$$

$$D(A) = \left\{ x \in X \mid \frac{d}{d\zeta}\mathcal{H}x \in X, W \begin{bmatrix} \mathcal{H}(b)x(b) \\ \mathcal{H}(a)x(a) \end{bmatrix} 0 \right\}$$

W = full rank matrix of size $n \times 2n$, $W_B = W \begin{bmatrix} P_1 & -P_1 \\ I & I \end{bmatrix}^{-1}$, $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$.

Theorem (Le Gorrec, Maschke & Zwart '05, J. Morris & Zwart '15)

A gen. a C_0-semigroup $\Leftrightarrow$ *matrix condition depending on W_B, P_1 and $\mathcal{H}$ is satisfied.*

A gen. a C_0-semigroup with $\|e^{At}\| \leq 1$ $\Leftrightarrow$ $W_B \Sigma W_B^T \geq 0$

$\Leftrightarrow$ $\langle Ax, x\rangle \leq 0$

A gen. a unitary C_0-group (i.e. e^{At} unitary) $\Leftrightarrow$ $W_B \Sigma W_B^T = 0$

$\Leftrightarrow$ $\langle Ax, x\rangle = 0$

Example: Wave equation

$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right]$$

$$\frac{\partial w}{\partial t}(0,t) = T(1)\frac{\partial w}{\partial \zeta}(1,t) = 0$$

$m \leq T(\zeta), \rho(\zeta) \leq M$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T(1)\frac{\partial w}{\partial \zeta}(1,t) \\ \frac{\partial w}{\partial t}(0,t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{=W} \begin{bmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)(0) \end{bmatrix}$$

$$W_B = W \left[\begin{smallmatrix} P_1 & -P_1 \\ I & I \end{smallmatrix}\right]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$

W_B has rank 2 and $W_B \Sigma W_B^T = 0$.Thus A generates a unitary group.

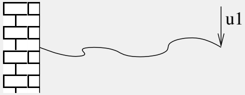
Port-Hamiltonian systems with inputs and outputs

We are interested in boundary controls and boundary observations.

$$\frac{\partial x}{\partial t}(\zeta,t) = \left(P_1 \frac{\partial}{\partial \zeta} + P_0\right)[\mathcal{H}x(t)]$$

$$u(t) = W_1 \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} \quad 0 = W_2 \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} \quad y(t) = W_C \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix}$$

Example: Wave equation



$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right]$$

$$u(t) = T(1)\frac{\partial w}{\partial \zeta}(1,t), \quad 0 = \frac{\partial w}{\partial t}(0,t)$$

$$y(t) = \frac{\partial w}{\partial t}(1,t)$$

Question: Is this a well-posed linear system?

Well-posedness of port-Hamiltonian systems

State space $X = L^2(a,b;\mathbb{R}^n)$ with norm $\|f\|_X^2 = \frac{1}{2}\int_a^b f(\zeta)^T\mathcal{H}(\zeta)f(\zeta)d\zeta$

Definition

The port-Hamiltonian system is called well-posed, if

- $Ax = P_1 \frac{d}{d\zeta}[\mathcal{H}x] + P_0[\mathcal{H}x]$ with domain

  $$D(A) = \{x \in X \mid (\mathcal{H}x)' \in X, \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} = 0\}$$

  is the generator of a C_0-semigroup on X.

- There are $t_0, m_{t_0} > 0$:

  $$\|x(t_0)\|_X^2 + \int_0^{t_0} \|y(t)\|^2 dt \le m_{t_0}\left[\|x(0)\|_X^2 + \int_0^{t_0}\|u(t)\|^2 dt\right]$$

Well-posedness of port-Hamiltonian systems

Let $W_B := \begin{bmatrix} W_{B,1} \\ W_{B,2} \end{bmatrix}$ be a full rank real matrix of size $n \times 2n$.

$P_1\mathcal{H}$ can be factorized as $P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta)$.

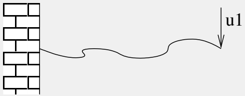
Assume: Δ, S are continuously differentiable

Theorem (Zwart, Le Gorrec, Maschke, Villegas '10)

If $Ax = \left(P_1 \frac{d}{d\zeta} + P_0\right) [\mathcal{H}x]$ generates a C_0-semigroup, then the port-Hamiltonian system is well-posed.

Remark: We even have a regular system.

Example: Wave equation



$$\begin{aligned} \frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right] \\ u(t) &= T(1)\frac{\partial w}{\partial \zeta}(1,t), \quad 0 = \frac{\partial w}{\partial t}(0,t) \\ y(t) &= \frac{\partial w}{\partial t}(1,t) \end{aligned}$$

$$P_1\mathcal{H} = \begin{bmatrix} 0 & T \\ \frac{1}{\rho} & 0 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma \\ \frac{1}{\rho} & \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix} \begin{bmatrix} \frac{1}{2\gamma} & \frac{\rho}{2} \\ -\frac{1}{2\gamma} & \frac{\rho}{2} \end{bmatrix} = S^{-1}\Delta S,$$

with $\gamma > 0$ und $\gamma^2 = \frac{T}{\rho}$.

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus: $W_B \Sigma W_B^T = 0$ and the controlled wave equation is well-posed.

Stability of port-Hamiltonian systems

Stability of abstract Cauchy systems

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

A is the generator of a C_0-semigroup $\left(e^{At}\right)_{t\geq 0}$ on X.

The abstract Cauchy system is exponentially stable $:\Leftrightarrow \exists\, M, \omega > 0$:

$$\|e^{At}\| \leq Me^{-\omega t}, \qquad t \geq 0.$$

Question

When is the Cauchy problem with

$$Ax = \left(P_1 \frac{d}{d\zeta} + P_0\right) [\mathcal{H}x]$$

$$D(A) = \{x \in X \mid (\mathcal{H}x)' \in X, \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \begin{bmatrix} (\mathcal{H}x)(b) \\ (\mathcal{H}x)(a) \end{bmatrix} = 0\}$$

exponentially stable?

Exponential stability of Port-Hamilton systems

Theorem (Villegas, Zwart, Le Gorrec & Maschke '09, J. & Zwart '12)

If there exists a constant $c > 0$ such that

$$\begin{aligned} &\langle Ax, x\rangle_X \le -c\|(\mathcal{H}x)(b)\|^2, \qquad x \in D(A) \\ \textit{or} \quad &\langle Ax, x\rangle_X \le -c\|(\mathcal{H}x)(a)\|^2, \qquad x \in D(A) \end{aligned}$$

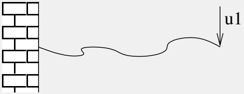
then the port-Hamiltonian system is exponentially stable.

Note: $2\langle Ax, x\rangle = (\mathcal{H}x)^T(b)P_1(\mathcal{H}x)(b) - (\mathcal{H}x)^T(a)P_1(\mathcal{H}x)(a)$

Sufficient condition

$W_B \Sigma W_B^T > 0 \Rightarrow$ port-Hamiltonian system is exponentially stable

Example: Wave equation



$$\frac{\partial^2 w}{\partial t^2}(\zeta,t) = \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right]$$

$$u(t) = T(1)\frac{\partial w}{\partial \zeta}(1,t), \quad 0 = \frac{\partial w}{\partial t}(0,t)$$

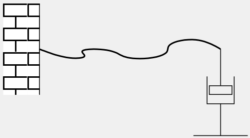
$$y(t) = \frac{\partial w}{\partial t}(1,t)$$

Question: Is the system exponentially stable?

No

We showed that the PDE generates a unitary group and thus the system is not exponentially stable.

Example: Wave equation with damper



$$
\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(\zeta,t) &= \frac{1}{\rho(\zeta)}\frac{\partial}{\partial \zeta}\left[T(\zeta)\frac{\partial w}{\partial \zeta}(\zeta,t)\right] \\
T(1)\frac{\partial w}{\partial \zeta}(1,t) &= -k\frac{\partial w}{\partial t}(1,t), \quad k>0 \\
0 &= \frac{\partial w}{\partial t}(0,t)
\end{aligned}
$$

$$
W_B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & k & k & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad W_B \Sigma W_B^T = \begin{bmatrix} 2k & 0 \\ 0 & 0 \end{bmatrix}.
$$

Sufficient condition on $W_B \Sigma W_B^T$ cannot be used.

Sufficient condition

$W_B \Sigma W_B^T > 0 \Rightarrow$ port-Hamiltonian system is exponentially stable

Example: Wave equation with damper

We have

$$2\langle x, Ax\rangle = \frac{\partial w}{\partial t}(1)T(1)\frac{\partial w}{\partial \zeta}(1) - \frac{\partial w}{\partial t}(0)T(0)\frac{\partial w}{\partial \zeta}(0) = -k\left(\frac{\partial w}{\partial t}(1)\right)^2$$

and

$$\|(\mathcal{H}x)(1)\|^2 = \left(\frac{\partial w}{\partial t}(1)\right)^2 + \left(T(1)\frac{\partial w}{\partial \zeta}(1)\right)^2 = (k^2+1)\left(\frac{\partial w}{\partial t}(1)\right)^2$$

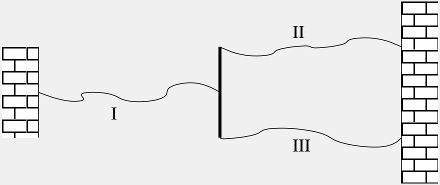
$$\Rightarrow \quad \langle x, Ax\rangle \leq -\frac{k}{2+2k^2}\|(\mathcal{H}x)(1)\|^2$$

and thus the feedback system is exponentially stable.

Conclusions

What we have done...

- We have formulated partial differential equations with boundary control and boundary observation as port-Hamiltonian systems
- Well-posedness and stability is guaranteed by a simple matrix test.
- It is easy to study coupled systems



Further results

- Characterisation of semigroup generation. (J. Morris, Zwart '15)
- Well-posedness for port-Hamiltonian systems with dissipation, that is, parabolic equations. (Augner, J. Laasri '15)
- $\frac{\partial x}{\partial t} = \left(P_2 \frac{\partial^2}{\partial^2 \zeta} + P_1 \frac{\partial}{\partial \zeta} + P_0\right) [\mathcal{H}x]$, for example Schrödinger and Euler-Bernoulli beam equations:
  Characterization of contr. semigr. (Le Gorrec, Zwart, Maschke '05)
  Characterization of stability. (Augner, J. '14)
- Port-Hamiltonian systems coupled with (linear or nonlinear) ODE
  Well-posedness and stability (Augner, J '14, Augner '15)
- Characterization of well-posedness of the wave equation in $\mathbb{R}^n$ (Kurula, Zwart '15)

Open Problems

Approximation

For controller design only a good approximation of the input-output behaviour is needed.
Many different types of systems approximations have been designed; i.e. balanced truncation, LQG-balancing, H_∞-approximation.
These controllers are robust, thus although designed for the approximations, they perform well on the original system.

Hyperbolic PDEs are hard to approx. due to high frequency effects.

Approximation of Port-Hamiltonian systems

Approximation by port-Hamiltonian systems:
R. Pasumarthy, V.R. Ambati, and A. van der Schaft '12
M. Seslija, J.M.A. Scherpen, A.J. van der Schaft '14
T. Voß and S. Weiland '11
The underlying structure is approximated.
Open Problem: Analysis of the convergence.

Thanks for your attention!