Recovering Structured Signals from Noisy Measurements: Where Least-Squares Meets Compressed Sensing

Babak Hassibi

joint work with
Samet Oymak and Christos Thrampoulidis

California Institute of Technology

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Outline

- **Introduction**
  - least-squares
- **Noiseless Structured Signal Recovery**
  - compressed sensing
  - convex relaxation
- **Phase Transitions**
  - escape-through-mesh, Gaussian widths, statistical dimension
  - non-uniform sparsity, low rank matrix recovery
  - connection to denoising
- **Structured Signal Recovery in Noise**
  - generalized LASSO
  - formulae for the squared error
- **Conclusion**

Least-Squares

Suppose we are confronted with the *noisy* measurements:

$$y = Ax + z,$$

where $A \in \mathcal{R}^{m \times n}$ is the measurement matrix, $y \in \mathcal{R}^m$ is the measurement vector, $x \in \mathcal{R}^n$ is the unknown desired signal, and $z \in \mathcal{R}^n$ is the unknown noise vector.

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A popular method for recovering x, is the least-squares criterion

$$\min_x \|y - Ax\|_2^2,$$

whose solution is famously given by

$$\hat{x} = \left(A^T A\right)^{-1} A^T y.$$

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$$\frac{\|x - \hat{x}\|_2^2}{m\sigma^2} \to \frac{n}{m - n}.$$

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Note that

$$\frac{n}{m-n} \leq \left(\frac{\sqrt{n}}{\sqrt{m}-\sqrt{n}}\right)^2 \leq \left(\frac{\sqrt{m}}{\sqrt{m}-\sqrt{n}}\right)^2.$$

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- The goal of compressed sampling is to perform sampling and sensing simultaneously

Noisy Compressed Sensing

Consider a "desired" signal $x \in \mathcal{R}^n$, which is k-sparse, i.e., has only $k < n$ (often $k \ll n$) non-zero entries. Suppose we make m noisy measurements of x using the $m \times n$ measurement matrix A to obtain

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This would give the normalized square errors:

$$\frac{k}{m-k} \quad , \quad \left(\frac{\sqrt{k}}{\sqrt{m} - \sqrt{k}} \right)^2 \quad , \quad \left(\frac{\sqrt{m}}{\sqrt{m} - \sqrt{k}} \right)^2 .$$

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There are also problems (such as low rank matrix recovery) where it is not possible to enumerate all structured signals.

LASSO

The LASSO algorithm was introduced by Tibshirani in 1996:

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where $\lambda \geq 0$ is a regularization parameter.

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Generalized LASSO

The generalized LASSO algorithm can be used to enforce other types of structures

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Often the value $f(\cdot)$ at the *true* x_0 is known a priori. In this case, we can alternatively solve

$$\begin{array}{ll} \min_x & \|y - Ax\|_2^2 \\ \text{subject to} & f(x) \leq f(x_0) \end{array}.$$

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Turns out *we can*. But to do so, we need to tell an earlier story....

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- Suppose each set of m columns of A are linearly independent. Then we can always find x uniquely from y—*via exhaustive search of* $\binom{n}{k}$ *systems of linear equations*—if $m > k$

But can we do this more efficiently? And for what values of m?

l_1 Optimization

The seminal work of Candes, Tao and Donoho has shown that under certain conditions the ℓ_1 optimization

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- Candes and Tao showed that if A satisfies certain *restricted isometry* conditions, then ℓ_1 optimization works for small enough k
  - gives "order optimal", but very loose bounds
- Necessary and sufficient conditions can be developed to obtain sharp bounds on k (Donoho-Tanner, Xu-H, Stojnic, Chandrasekaran-Parrilo-Willsky, Oymak-H)
  - we will get into this

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  - tightness of Gaussian widths Stojnic, 2013 (for ℓ_1), Amelunxen-McCoy-Tropp, 2013 (more generally)

Phase Transitions for Convex Relaxation

We will consider a general framework.

Consider a structured signal x_0, with a structure-capturing atomic norm $f(\cdot) = \|\cdot\|$. We have access to *linear measurements* $y = \mathcal{A}(x_0) \in R^m$, and would like to know when we can recover the signal x_0 from the convex problem

$$\min \|x\| \quad \text{subject to } \mathcal{A}(x) = \mathcal{A}(x_0)?$$

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- For sparse signals we have the ℓ_1 norm; for nonuniform sparse signals the weighted ℓ_1 norm; for low rank matrices the nuclear norm

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Consider a structured signal x_0, with a structure-capturing atomic norm $f(\cdot) = \|\cdot\|$. We have access to *linear measurements* $y = \mathcal{A}(x_0) \in R^m$, and would like to know when we can recover the signal x_0 from the convex problem

$$\min \|x\| \quad \text{subject to } \mathcal{A}(x) = \mathcal{A}(x_0)?$$

- For sparse signals we have the ℓ_1 norm; for nonuniform sparse signals the weighted ℓ_1 norm; for low rank matrices the nuclear norm

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$$\mathcal{N}(\mathcal{A}) \cap \mathcal{U}(x_0) = \{0\}.$$

A Bit of Geometry: Subgradients and the Polar Cone

Note that $\mathcal{N}(\mathcal{A})$ is a linear subspace and that therefore the condition can be rewritten as

$$\mathcal{N}(\mathcal{A}) \cap \text{cone}(\mathcal{U}(x_0)) = \{0\}.$$

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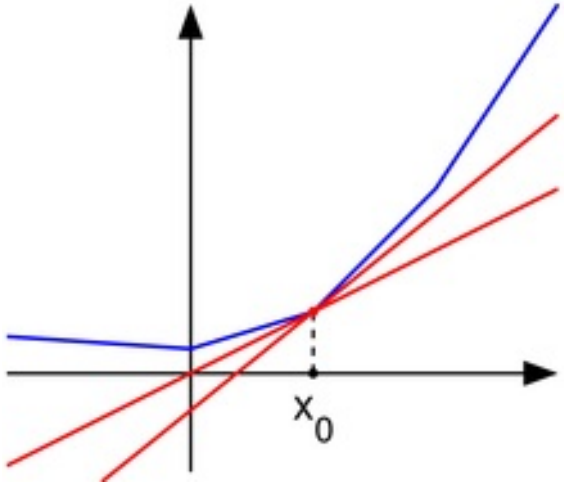
$$\mathcal{N}(\mathcal{A}) \cap \text{cone}(\mathcal{U}(x_0)) = \{0\}.$$

We can characterize $\text{cone}(\mathcal{U}(x_0))$ through the subgradient of the convex function $\|\cdot\|$:

$$\partial\|x_0\| = \{v | v^T(x - x_0) \leq \|x\| - \|x_0\|, \forall x\}.$$



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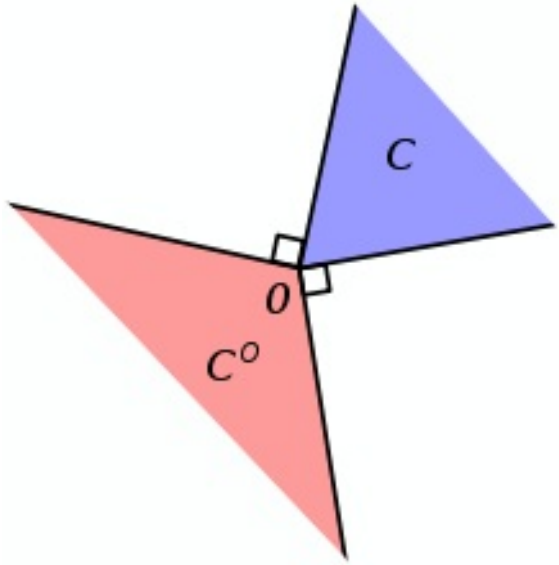
But this is simply the *polar cone* of $\partial\|x_0\|$.



Thus, we can recover x_0 from the convex problem iff:

$$\mathcal{N}(\mathcal{A}) \cap (\partial\|x_0\|)^O = \{0\}.$$

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- However, checking the condition $\mathcal{N}(\mathcal{A}) \cap (\partial \|x_0\|)^O = \{0\}$ for a *specific* $\mathcal{A}$ is difficult.
- Therefore the focus has been on checking whether the condition holds for a *family* of random $\mathcal{A}$'s with high probability.

The Gaussian Measurement Ensemble

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- This makes the nullspace $\mathcal{N}(\mathcal{A})$ *rotationally-invariant*.
- The thresholds obtained from the Gaussian matrix ensemble are often *universal* and so determine the thresholds for other matrix ensembles.

Escape Through a Mesh

Theorem (Escape Through a Mesh - Gordon 1988)

Let C be a subset of unit sphere S^{n-1} in $\mathbb{R}^n$ and $\mathbf{g}$ be a vector with i.i.d. standard normal entries. Further, let H be an $n-m$ dimensional subspace distributed uniformly over Grassmannian w.r.t Haar measure. Then,

$$\mathbb{P}(H \cap C = \emptyset) \geq 1 - 3.5 \exp(-(\sqrt{m} - \frac{1}{4\sqrt{m}} - \omega(C))^2)$$

where $\omega(C)$ stands for the Gaussian width of C and is defined as:

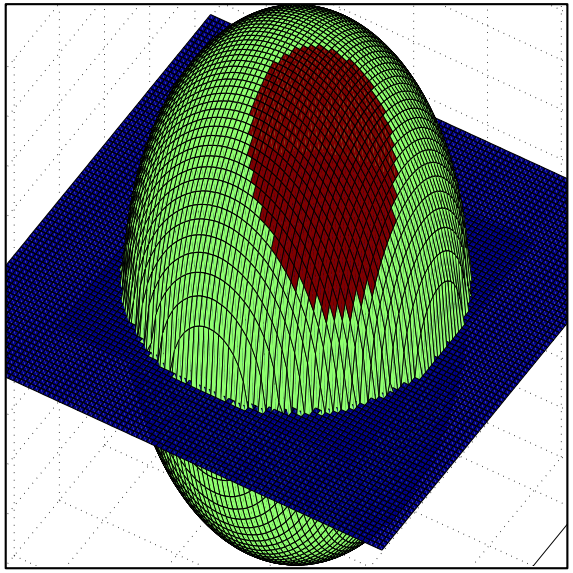
$$\omega(C) = \mathbb{E}[\sup_{\mathbf{v} \in C} \langle \mathbf{g}, \mathbf{v} \rangle]$$

Interpretation

- Question: When $\mathbf{A}$ has i.i.d. Gaussian entries what is the chance of:

$$\text{Null}(\mathbf{A}) \cap (\partial \|x_0\|)^O = \{0\}?$$

- Answer
  - Green ball: Unit sphere S^{n-1}.
  - Blue plane: Random subspace Null($\mathbf{A}$).
  - Red mesh: Undesired set $C = (\partial \|x_0\|)^O \cap S^{n-1}$.



Gaussian Width Calculation

- To ensure recovery: choose $m \geq \omega((\partial\|x_0\|)^O \cap S^{n-1})^2$
- It is critical to estimate the GW accurately

Lemma

Let $\partial\|\mathbf{x}_0\|$ be the subdifferential of $\|\cdot\|$ at $\mathbf{x}_0$ and $\mathbf{g}$ be a vector with i.i.d. standard normal entries. Then

$$\omega((\partial\|x_0\|)^O \cap S^{n-1}) = \mathbb{E}[dist(\mathbf{g}, \partial\|\mathbf{x}_0\|)] \approx \sqrt{\mathbb{E}[dist(\mathbf{g}, \partial\|\mathbf{x}_0\|)^2]}. \quad (0.1)$$

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Example: Nuclear Norm Minimization (NNM)

Nuclear norm: $\|X\|_\star = \sum_i \sigma_i(X)$

- Many applications: Netflix, minimal order controllers, graphical models, etc.
- Tightest convex relaxation to rank minimization problem

$$\min \|X\|_\star \quad \text{subject to} \quad \mathcal{A}(X) = y$$

- Closely related to ℓ_1 minimization
- Polynomial time algorithm (an SDP)
- Known to have noise robustness
- Unlike non-convex algorithms NNM can be made more complicated.
  - coupling with ℓ_1: Robust PCA
    $$\min \|L\|_\star + \lambda \|S\|_1 \quad \text{subject to} \quad X = L + S$$

Prior Work

Let $\mathbf{y} = \mathcal{A}(X) \in \mathbb{R}^m$, $X \in \mathbb{R}^{n \times n}$.
Number of samples for successful recovery is a useful measure of performance

- i.i.d. measurements
  - Recht, Fazel, Parrilo "Guaranteed minimum rank ..." (2008): $m = O(rnlog(n))$
  - Candes, Plan "Tight oracle bounds ..." (2010): $m = O(rn)$
  - First null space analysis: Recht, Xu, H "Null Space Conditions and ..." (2009)
- Observe entries at random (matrix completion)
  - Candes, Recht "Exact matrix completion via ..." (2009): $m = O(rn^{5/4}log(n))$.
  - Best known: $m = nrlog^2(n)$.
    - ★ Recht "A simpler approach ..." (2009)
    - ★ Gross "Recovering low rank matrices ..." (2011)

Strong and Weak Robustness

Let M^* denote optimal sol'n of "$\min \|X\|_\star \quad \text{subject to} \quad \mathbf{A}(X) = \mathbf{A}(M)$".

Definition (Strong robustness)

$\mathcal{A}(\cdot)$ is strongly robust with ϵ_0, r if the following holds for all M:

$$\|M - M^*\|_\star < \frac{2\|M - M^r\|_\star}{\epsilon_0}$$

$M - M^r$ is basically the tail of M.

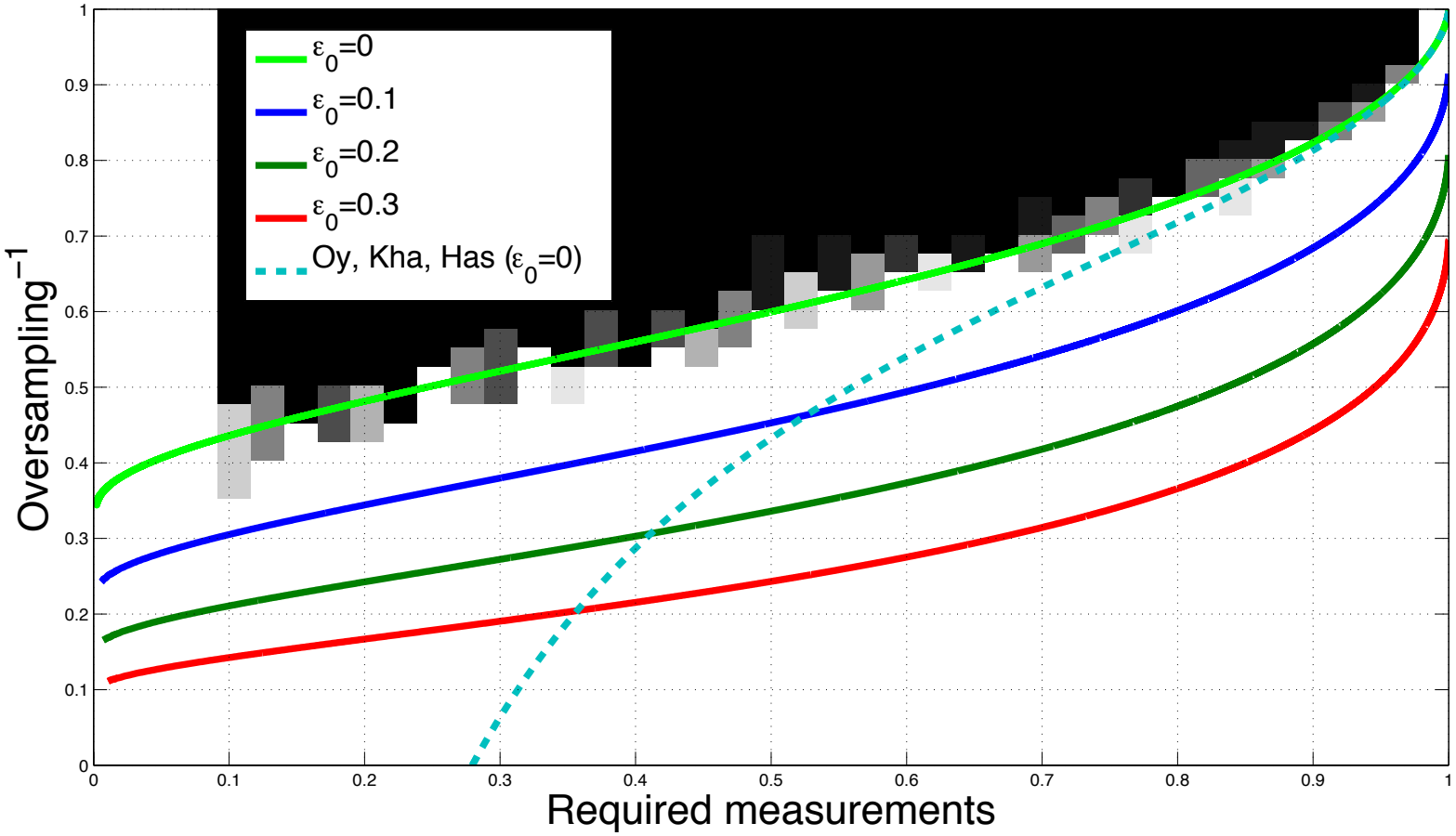
Definition (Weak robustness)

Given a fixed M, $\mathcal{A}(\cdot)$ is weakly robust with ϵ_0, r if the following holds:

$$\|M - M^*\|_\star < \frac{2\|M - M^r\|_\star}{\epsilon_0}$$

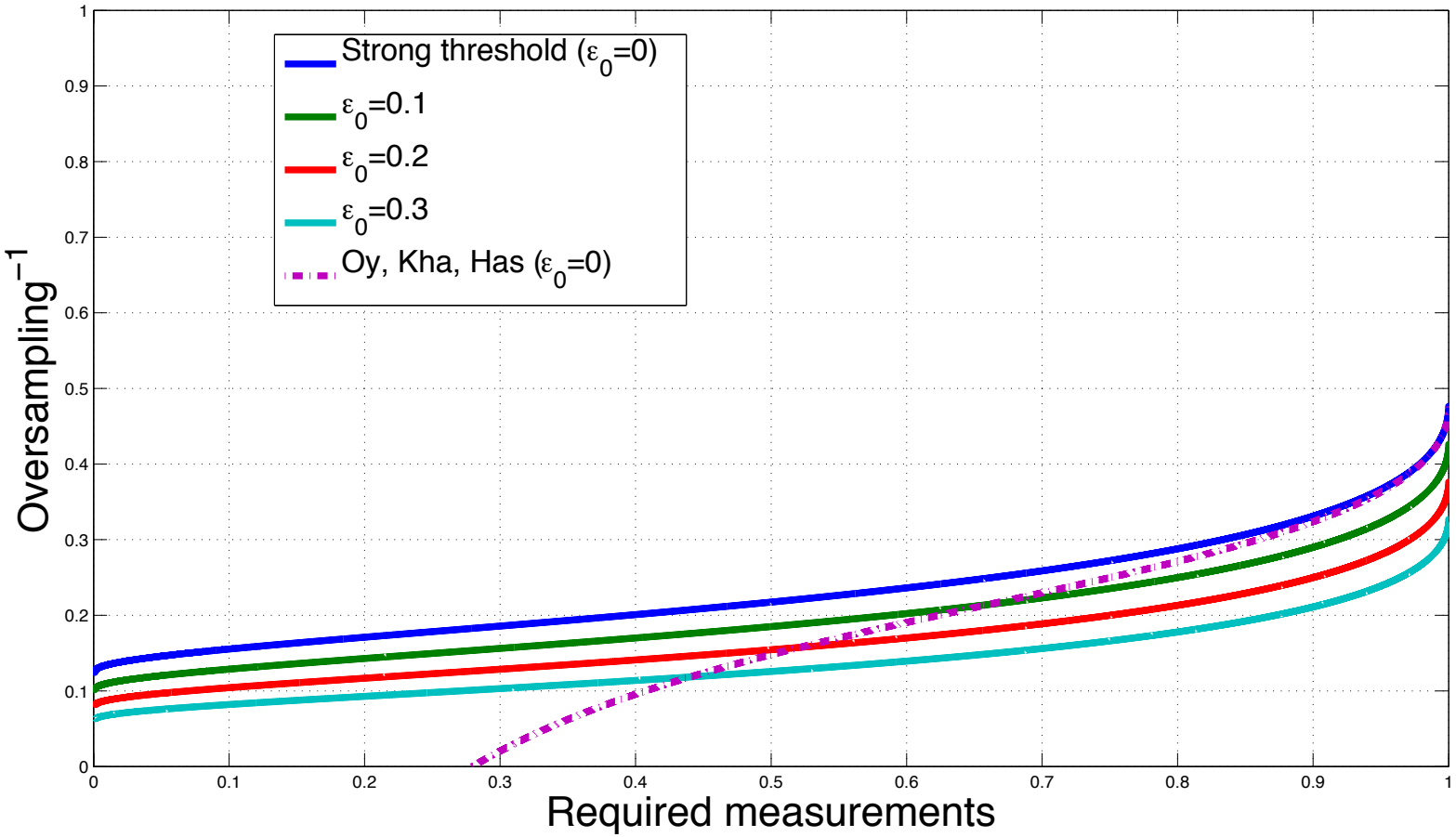
Weak recovery curves

Oversampling: How much measurement per degrees of freedom?



DOF $= r(2n - r)$

Strong recovery curves



Closed Form Results

Theorem (Closed Form Bounds)

Let dimensions be $n_1 \times n_2$, maximum rank be r (no tail). Then

$$\begin{aligned} &\textit{for strong recovery:} \quad m \geq 4(\sqrt{n_1} + \sqrt{n_2})^2 r \\ &\textit{for weak recovery:} \quad m \geq 2(n_1 + \sqrt{n_1 n_2} + n_2) r \end{aligned}$$

is sufficient.

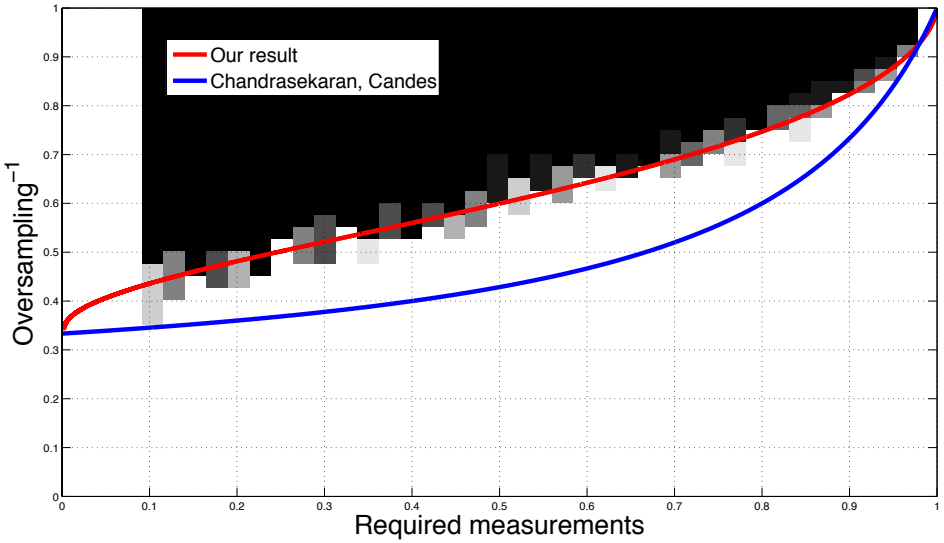
Theorem (Closed Form Bounds)

In the square case:

$$\begin{aligned} &\textit{for strong recovery:} \quad m \geq 16nr \\ &\textit{for weak recovery:} \quad m \geq 6nr \end{aligned}$$

Recent and Simultaneous Work

- Chandrasekaran, Recht, Parrilo, and Willsky "The convex geometry of linear inverse problems" (2010)
- Candes, Recht "Simple bounds for low complexity model construction" (2011)



Some Other Examples

- n dimensional vectors that are k-sparse:

$$f(x) = \|x\|_1 \quad \text{and} \quad \omega^2 \leq 2k \log \frac{2k}{n}$$

- $n \times n$ matrices that are rank r:

$$f(x) = \|x\|_\star \quad \text{and} \quad \omega^2 \leq 3r(2n - r)$$

- qb dimensional vectors that have k non-zero blocks of size b:

$$f(x) = \|x\|_{1,2} \quad \text{and} \quad \omega^2 \leq 4k \left(b + log \frac{q}{k}\right)$$

Relation Between Compressed Recovery and Denoising

- **CS problem:** Recover a signal from underdetermined linear observations.

$$\mathbf{y} = \mathbf{A}\mathbf{x}_0 \in \mathbb{R}^m, \ \mathbf{x}_0 \in \mathbb{R}^n, \ \mathbf{A} \sim \text{i.i.d. } \mathcal{N}(0,1)$$

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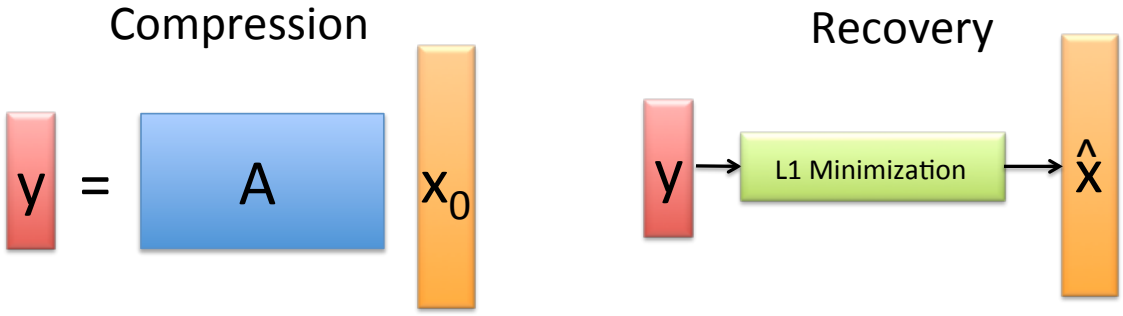

- We will focus on convex $f(\cdot)$

Denoising Problem

- **Denoising problem:** Estimate a signal corrupted by additive noise.

$$\mathbf{y} = \mathbf{x}_0 + \mathbf{z}, \ \mathbf{x}_0 \in \mathbb{R}^n, \ \mathbf{z} \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2)$$

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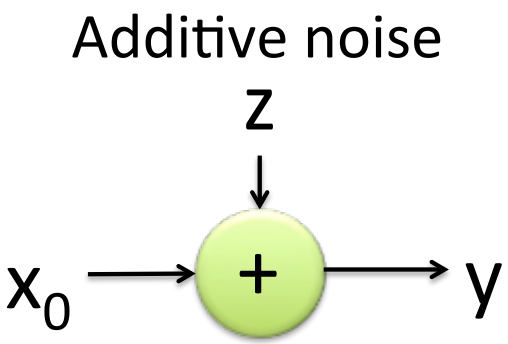
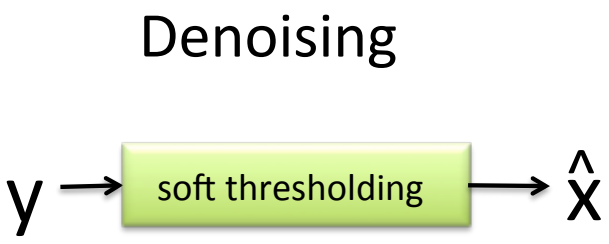




Performance Criteria

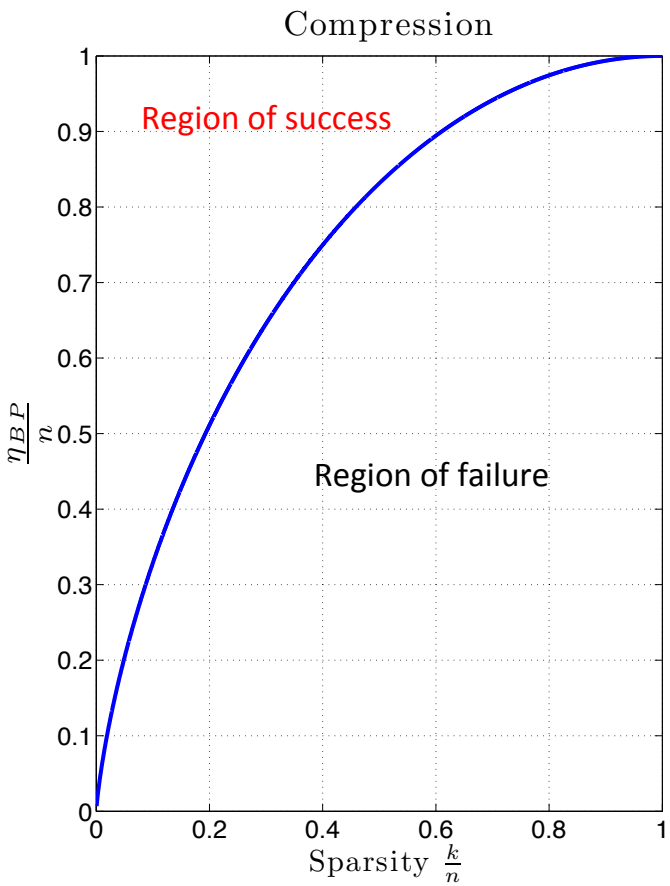
- **CS problem:** The smallest number $\eta_{BP}(\mathbf{x}_0)$ s.t. $m = \eta_{BP}(\mathbf{x}_0)$ measurements are sufficient for recovery of $\mathbf{x}_0$ via (BP) w.h.p. where $\mathbf{y} = \mathbf{A}\mathbf{x}_0 \in \mathbb{R}^m$.
- **Denoising problem:** Tune λ optimally to minimize normalized estimation error.

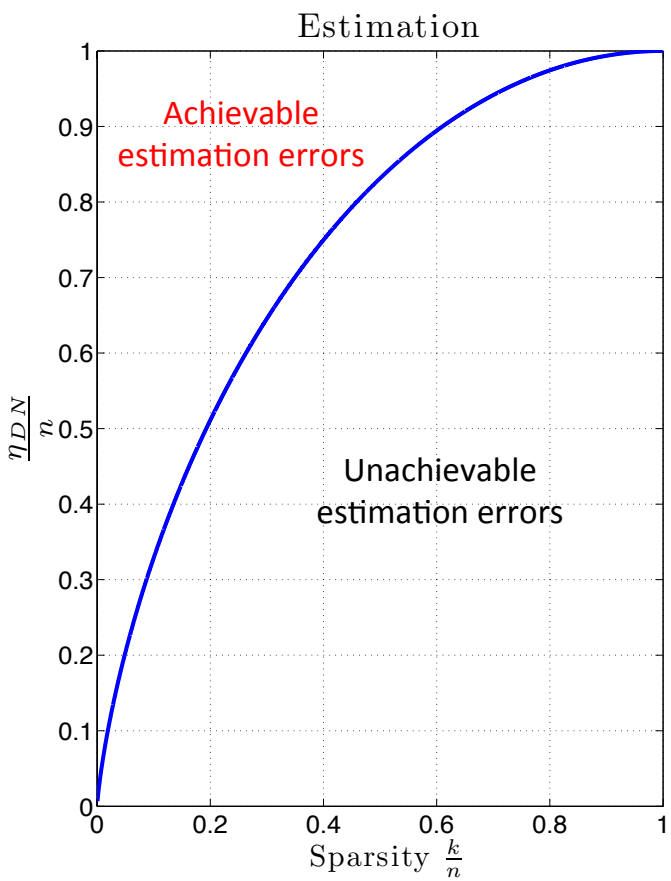
$$\eta_{DN}(\mathbf{x}_0) = \lim_{\sigma \to 0} \inf_{\lambda \geq 0} \frac{\mathbb{E}[\|\mathbf{x}_0 - \mathbf{x}^*(\lambda, \mathbf{z})\|_2^2]}{\sigma^2} \tag{1.1}$$

where $\mathbf{x}^*(\lambda, \mathbf{z})$ is the minimizer of LASSO:

$$\mathbf{x}^*(\lambda, \mathbf{z}) = \arg\min_{\mathbf{x}} \lambda f(\mathbf{x}) + \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|_2^2 \tag{1.2}$$

ℓ_1 Phase Transitions





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- Recently, based on extensive empirical evidence, Donoho, Montanari and Johnstone have proposed that

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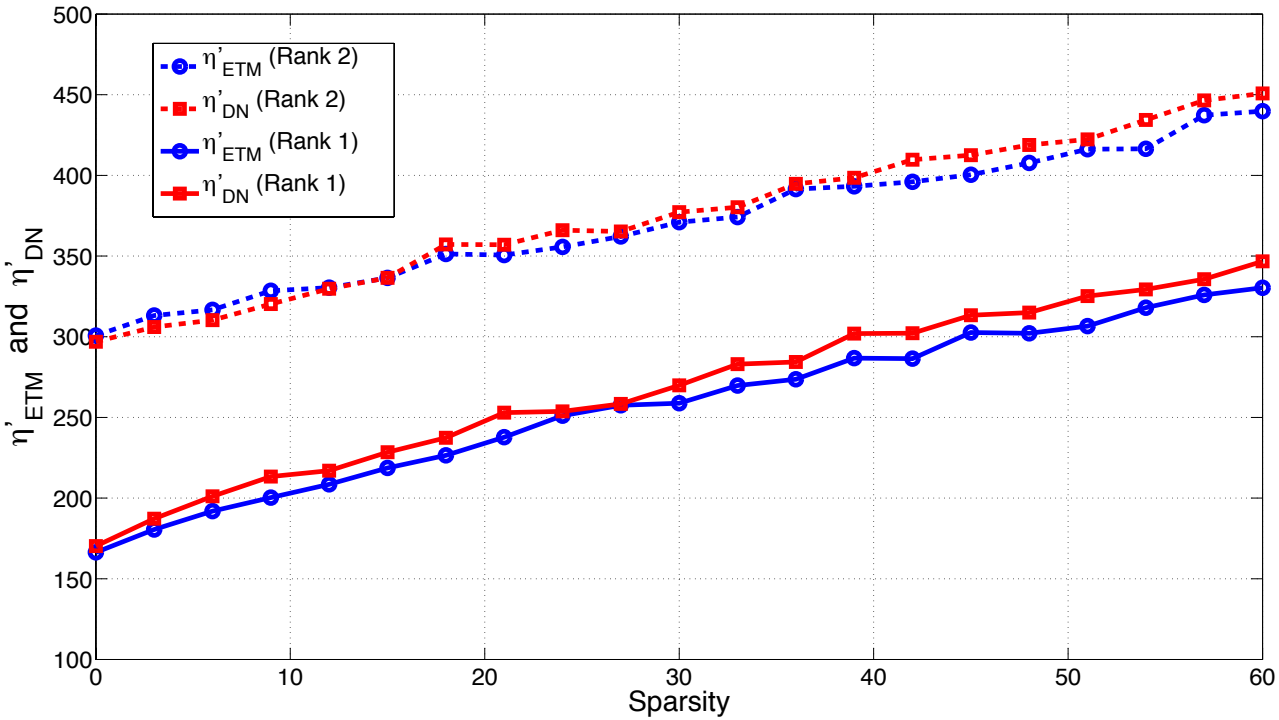
  - "Accurate Prediction of Phase Transitions in Compressed Sensing ..." (2011)
- In Oymak-H using the escape-through-mesh framework we show that $\eta_{BP} = \eta_{DN} = \omega^2$, the statistical dimension, for a general convex $f(\cdot)$.

Low Rank plus Sparse Signals

$$\mathbf{X} \text{ is LPS and } f(\mathbf{X}) = \min_{\mathbf{L}} \|\mathbf{L}\|_\star + \Theta\|\mathbf{X} - \mathbf{L}\|_1$$

sparsity: $0 \to 60$, rank: $1 \to 2$



Observation: $\eta_{DN} \approx \eta_{ETM}$.

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$\mathbf{x}_0 \in \mathbf{R}^n$: *structured* signal of interest (sparse, low-rank, etc.)

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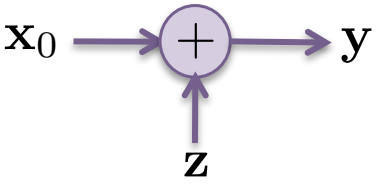

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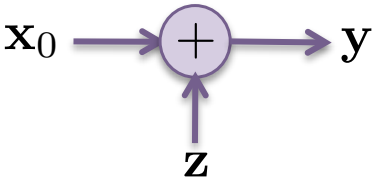

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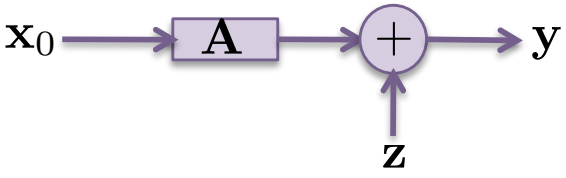
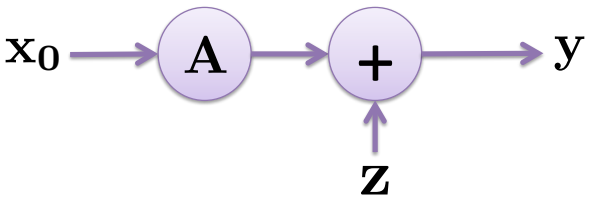
3. Noisy CS (LASSO) can be seen as a "merger" of 1 and 2:



$$\min_{\mathbf{x}} \left\{ \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda f(\mathbf{x}) \right\}$$

The Generalized LASSO

Observe



Solve $\min_{\mathbf{x}} \lambda f(\mathbf{x}) + \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2$

- $f(\mathbf{x}) = \|\mathbf{x}\|_1$: R. Tibshirani "Regression shrinkage and selection via the lasso", '96., Chen, Donoho, Saunders'98
- $f(\mathbf{X}) = \|\mathbf{X}\|_*$: Koltchinskii'10, Negahban'12, Candes'11 and more.

✓ $f(\mathbf{x})$ = any *convex* function of $\mathbf{x}$: **"Generalized LASSO"**

The Squared Error of Generalized LASSO

Consider the generalized LASSO:

$$\min_{\mathbf{x}} \left\{ \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \lambda f(\mathbf{x}) \right\},$$

then we have been able to show:

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then we have been able to show:

- for z iid $N(0, \frac{1}{m})$ Gaussian

$$\min_{\lambda} \frac{\|x_0 - \hat{x}\|_2^2}{\|z\|_2^2} \rightarrow \frac{\omega^2}{m - \omega^2}$$

- for z arbitrary, but independent of A:

$$\min_{\lambda} \frac{\|x_0 - \hat{x}\|_2^2}{\|z\|_2^2} \rightarrow \leq \left(\frac{\omega}{\sqrt{m} - \omega} \right)^2$$

- for the worst-case z, chosen with knowledge of A:

$$\min_{\lambda} \frac{\|x_0 - \hat{x}\|_2^2}{\|z\|_2^2} \rightarrow \left(\frac{\sqrt{m}}{\sqrt{m} - \omega} \right)^2$$

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These are the same formulae obtained for standard least-squares, except that the *ambient dimension* , n, has been replaced by the *statistical dimension*, ω^2.
These results are also true for the constrained LASSO

$$\min_x \|y - Ax\|_2^2 \quad \text{subject to } f(x) \leq f(x_0).$$

For example,

- for n-dimensional k-sparse signals and ℓ_1 minimization, we have the bounds

$$\frac{2k \log \frac{2n}{k}}{m - 2k \log \frac{2n}{k}} \quad , \quad \left(\frac{\sqrt{2k \log \frac{2n}{k}}}{\sqrt{m} - \sqrt{2k \log \frac{2n}{k}}} \right)^2 \quad , \quad \left(\frac{\sqrt{m}}{\sqrt{m} - \sqrt{2k \log \frac{2n}{k}}} \right)^2$$

The Squared Error of Generalized LASSO

- for $n \times n$-dimensional rank r matrices and nuclear norm minimization, we have the bounds

$$\frac{3r(2n-r)}{m-3r(2n-r)}\,,\quad \left(\frac{\sqrt{3r(2n-r)}}{\sqrt{m}-\sqrt{3r(2n-r)}}\right)^2\,,\quad \left(\frac{\sqrt{m}}{\sqrt{m}-\sqrt{3r(2n-r)}}\right)^2$$

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- for qb-dimensional k-block sparse vectors and mixed ℓ_1/ℓ_2 minimization we have the bounds

$$\frac{4k(b+\log\frac{q}{k})}{m-4k(b+\log\frac{q}{k})}\,,\quad \left(\frac{\sqrt{4k(b+\log\frac{q}{k})}}{\sqrt{m}-\sqrt{4k(b+\log\frac{q}{k})}}\right)^2\,,\quad \left(\frac{\sqrt{m}}{\sqrt{m}-\sqrt{4k(b+\log\frac{q}{k})}}\right)$$

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- These results required either knowledge of the *optimal* λ, or of $f(x_0)$.
- But what if we do not have these?
- Can we still give formulae for an arbitrary $\lambda \geq 0$?

Three Versions of the LASSO Problem

Observe $\mathbf{y} = \mathbf{A}\mathbf{x}_0 + \mathbf{z}$, where $\mathbf{A} \sim \mathcal{N}(0, \frac{1}{m}\mathbf{I}_{m\times n})$ $\mathbf{z} \sim \mathcal{N}(0, \sigma^2\mathbf{I}_m)$

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1. Constrained LASSO

$$\mathbf{x}_c^* = \arg\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \quad \text{subject to} \quad f(\mathbf{x}) \leq f(\mathbf{x}_0)$$

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$$\mathbf{x}_{\ell_2}^* = \arg\min_{\mathbf{x}} \{\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 + \lambda f(\mathbf{x})\}$$

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Quantity of Interest:

$$\boxed{\text{Normalized Squared Error} := \frac{\|\mathbf{x}_{\text{LASSO}}^* - \mathbf{x}_0\|_2^2}{\|z\|_2^2}}$$

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  - *we rely on this framework*

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- Simple recipe for **optimal tuning** of the regularization parameter λ
- **Converse results**: When does robust estimation of $\mathbf{x}_0$ fail? Connection to phase transitions of noiseless CS, to statistical dimension, and to similar formulae in standard least-squares.
- **Extend** the results from z iid $N(0, \sigma^2)$ to **arbitrary** z and **worst-case** z.

First-Order Approximation

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Approximated LASSO Problem:

$$\min_{\mathbf{w}} \left\{ \|\mathbf{A}\mathbf{w} - \mathbf{z}\|_2 + \sup_{\mathbf{s} \in \lambda \partial f(\mathbf{x}_0)} \mathbf{s}^T \mathbf{w} \right\}$$

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- For the Constrained Problem, just replace $\lambda \partial f(\mathbf{x}_0)$ with $\text{cone}(\partial f(\mathbf{x}_0))$.

Why $\sigma \to 0$?

- **Precise Formulas**
  - First-order approximation is tight $\implies$ Perform the analysis on the more tractable Approximate LASSO, instead.
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- **Precise Formulas**
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  - Precise analysis for arbitrary σ impossible with f.o. approximation.
- **Worst-case error scenario**
  - Worst case NSE of the LASSO is achieved for $\sigma \to 0$.
  - Our formulae *upper bound* the NSE for *arbitrary* values of the noise variance.

Gordon's Lemma

- Introduced by Gordon in '88. Compares two centered Gaussian processes:

Lemma (Gordon)

Let $\mathbf{G} \in \mathbf{R}^{\mathbf{m}\times\mathbf{n}}, g \in \mathbf{R}, \mathbf{g} \in \mathbf{R}^{\mathbf{m}}, \mathbf{h} \in \mathbf{R}^{\mathbf{n}}$, *all having entries i.i.d.* $\mathcal{N}(0,1)$ *and being independent of each other. Also, let* $\mathcal{S} \subset \mathbb{R}^n$ *be arbitrary set and* $\psi : \mathcal{S} \to \mathbb{R}$ *be arbitrary function. Then, for all choices of* $c \in \mathbb{R}$,

$$\mathbb{P}\left(\min_{\mathbf{x}\in\mathcal{S}} \{\|\mathbf{G}\mathbf{x}\|_2 + \|\mathbf{x}\|_2 g - \psi(\mathbf{x})\} \geq c\right) \geq \mathbb{P}\left(\min_{\mathbf{x}\in\mathcal{S}} \left\{\|\mathbf{x}\|_2\|\mathbf{g}\|_2 - \mathbf{h}^T\mathbf{x} - \psi(\mathbf{x})\right\} \geq c\right)$$

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Apply (a slight modification) to Approximated LASSO:

$$\mathbb{P}\left(\min_{\mathbf{w}} \left\{\|\mathbf{A}\mathbf{w} - \mathbf{z}\|_2 + \sup_{\mathbf{s}\in\lambda\partial f(\mathbf{x}_0)} \mathbf{s}^T\mathbf{w}\right\} \geq c\right)$$

$$\geq 2\,\mathbb{P}\left(\min_{\mathbf{w}} \left\{\sqrt{\|\mathbf{w}\|_2^2 + \sigma^2}\|\mathbf{g}\|_2 - \mathbf{h}^T\mathbf{w} + \sup_{\mathbf{s}\in\lambda\partial f(\mathbf{x}_0)} \mathbf{s}^T\mathbf{w}\right\} \geq c\right) - 1$$

After Gordon's Lemma: Deterministic Analysis

Key Optimization

$$\mathcal{L}(\mathbf{g},\mathbf{h}) = \min_{\mathbf{w}} \left\{ \sqrt{\|\mathbf{w}\|_2^2 + \sigma^2}\|\mathbf{g}\|_2 - \mathbf{h}^T\mathbf{w} + \sup_{\mathbf{s}\in\lambda\partial f(\mathbf{x}_0)} \mathbf{s}^T\mathbf{w} \right\}$$

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- Solve:

$$\|\mathbf{w}^*\|_2^2 = (\alpha^*)^2 = \sigma^2 \frac{\mathrm{dist}^2(\mathbf{h},\lambda\partial f(\mathbf{x}_0))}{\|\mathbf{g}\|_2^2 - \mathrm{dist}^2(\mathbf{h},\lambda\partial f(\mathbf{x}_0))} \quad , \quad \mathcal{L}(\mathbf{g},\mathbf{h}) = \sigma\sqrt{\|\mathbf{g}\|_2^2 - \mathrm{dist}^2(\mathbf{h},\lambda\partial f(\mathbf{x}_0))}$$

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- *Easy*: $\mathbf{h} \sim \mathcal{N}(0,\mathbf{I}_n)$, $\mathbf{g} \sim \mathcal{N}(0,\mathbf{I}_m)$ and independent!
- Basic *Concentration* Arguments:

$$\text{dist}^2(\mathbf{h},\lambda\partial f(\mathbf{x}_0)) \approx \underbrace{\mathbb{E}_{\mathbf{h}}\left[\text{dist}^2(\mathbf{h},\lambda\partial f(\mathbf{x}_0))\right] := \mathbf{D}_f(\mathbf{x}_0,\lambda)}_{\text{"Gaussian Squared Distance"}}$$

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$$\text{dist}^2(\mathbf{h},\lambda\partial f(\mathbf{x}_0)) \approx \underbrace{\mathbb{E}_{\mathbf{h}}\left[\text{dist}^2(\mathbf{h},\lambda\partial f(\mathbf{x}_0))\right] := \mathbf{D}_f(\mathbf{x}_0,\lambda)}_{\text{"Gaussian Squared Distance"}}$$

- Apply to Deterministic results:

$$\mathcal{L}(\mathbf{g},\mathbf{h}) = \sigma\sqrt{\|\mathbf{g}\|_2^2 - \text{dist}^2(\mathbf{h},\lambda\partial f(\mathbf{x}_0))} \approx \boxed{\sigma\sqrt{m - \mathbf{D}_f(\mathbf{x}_0,\lambda)}} \implies$$

High-Probability Lower Bound for LASSO cost!

$$\frac{\|\mathbf{w}^*\|_2^2}{\|z\|_2^2} = \frac{\text{dist}^2(\mathbf{h},\lambda\partial f(\mathbf{x}_0))}{\|\mathbf{g}\|_2^2 - \text{dist}^2(\mathbf{h},\lambda\partial f(\mathbf{x}_0))} \approx \boxed{\frac{\mathbf{D}_f(\mathbf{x}_0,\lambda)}{m - \mathbf{D}_f(\mathbf{x}_0,\lambda)}} \xrightarrow{???}$$

Normalized Square Error of the LASSO !

Synopsis of the Technical Framework

$$\mathcal{F}_{\ell_2}(\mathbf{A},\mathbf{z}) = \min_{\mathbf{w}} \left\{ \|\mathbf{A}\mathbf{w}-\mathbf{z}\|_2 + \sup_{\mathbf{s}\in\lambda\partial f(\mathbf{x}_0)} \mathbf{s}^T\mathbf{w} \right\}$$

1. Gordon's Lemma to $\mathcal{F}_{\ell_2}(\mathbf{A},\mathbf{z})$ to find a *high-probability* **lower bound** for it. We showed that!
2. Gordon's Lemma to the *dual* of $\mathcal{F}_{\ell_2}(\mathbf{A},\mathbf{z})$ to find a *high-probability* **upper bound** for it
3. Lower and Upper Bounds **match** $\Longrightarrow$ $\boxed{\mathcal{F}_{\ell_2}(\mathbf{A},\mathbf{z}) \approx \sigma\sqrt{m-\mathbf{D}_f(\mathbf{x}_0,\lambda)}}$
4. Assume $\|\mathbf{w}^*_{\ell_2}\|_2^2$ deviates from $\sigma^2\frac{\mathbf{D}_f(\mathbf{x}_0,\lambda)}{m-\mathbf{D}_f(\mathbf{x}_0,\lambda)}$. Then, Gordon's Lemma shows that optimization cost is *strictly larger* than $\sigma\sqrt{m-\mathbf{D}_f(\mathbf{x}_0,\lambda)}$.
5. Conclude $\boxed{\frac{\|\mathbf{w}^*_{\ell_2}\|_2^2}{\|z\|_2^2} \approx \frac{\mathbf{D}_f(\mathbf{x}_0,\lambda)}{m-\mathbf{D}_f(\mathbf{x}_0,\lambda)}}$

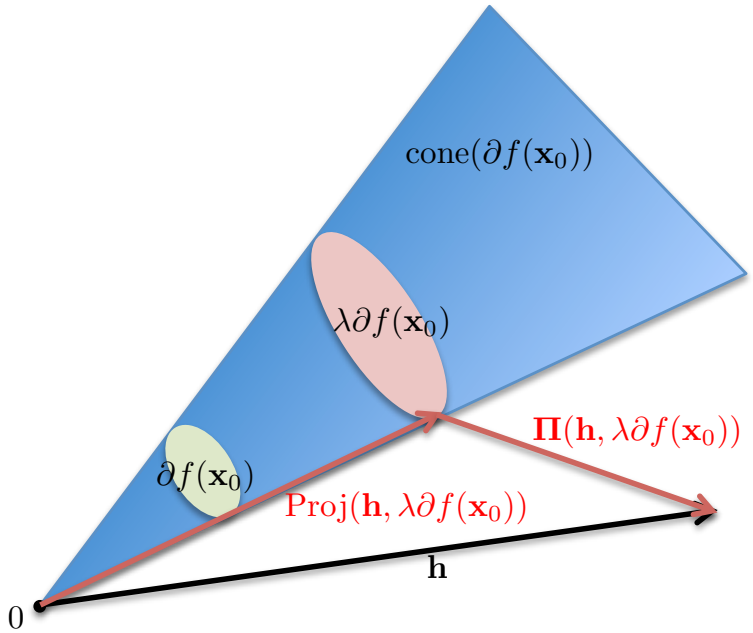
Gaussian Squared Distance and Related Quantities

$\text{dist}(\mathbf{h}, \lambda\partial f(\mathbf{x}_0)) := \|\Pi(\mathbf{h}, \lambda\partial f(\mathbf{x}_0))\|_2$

$\mathbf{D}_f(\mathbf{x}_0, \lambda) := \mathbb{E}_{\mathbf{h}}\left[\text{dist}^2(\mathbf{h}, \lambda\partial f(\mathbf{x}_0))\right]$

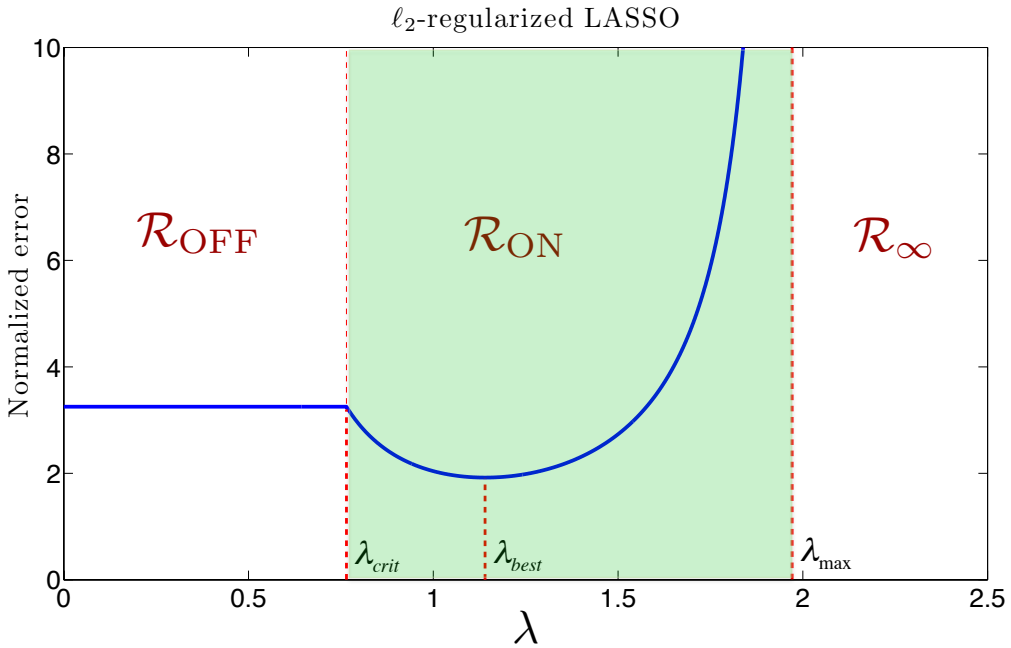
$\mathbf{\Delta}_f(\mathbf{x}_0) = \omega^2 := \mathbb{E}_{\mathbf{h}}\left[\text{dist}^2(\mathbf{h}, \text{cone}(\partial f(\mathbf{x}_0)))\right]$

$\mathbf{C}_f(\mathbf{x}_0, \lambda) := \mathbb{E}_{\mathbf{h}}\left[\langle \text{Proj}(\mathbf{h}, \lambda\partial f(\mathbf{x}_0)), \Pi(\mathbf{h}, \lambda\partial f(\mathbf{x}_0))\rangle\right]$



- $\mathbf{\Delta}_f(\mathbf{x}_0) = \omega^2$: minimum number of measurements required to success of the *Noiseless CS* (*Gaussian width* or *statistical dimension*)
- $\mathbf{D}_f(\mathbf{x}_0, \lambda)$: upper bounds the normalized squared error of the *simple denoiser*
- $\min_{\lambda \geq 0} \mathbf{D}_f(\mathbf{x}_0, \lambda) = \mathbf{\Delta}_f(\mathbf{x}_0)$
- $\mathbf{C}_f(\mathbf{x}_0, \lambda)$: Appears when upper bounding LASSO cost in Step 2

ℓ_2-LASSO: Regions of Operation



R_{OFF}

$$m - \mathbf{D}_f(\mathbf{x}_0, \lambda) \leq \mathbf{C}_f(\mathbf{x}_0, \lambda)$$

$$\mathbf{A}\mathbf{x}^* = \mathbf{y}$$

R_{ON}

$$m \geq \mathbf{D}_f(\mathbf{x}_0, \lambda)$$

$$m - \mathbf{D}_f(\mathbf{x}_0, \lambda) \geq \mathbf{C}_f(\mathbf{x}_0, \lambda)$$

Precise Bounds!

R_{∞}

$$m \leq \mathbf{D}_f(\mathbf{x}_0, \lambda)$$

Not Robust Recovery

ℓ_2-LASSO: R_{ON}

Theorem (ℓ_2-LASSO)

Assume,

- $\frac{\mathbf{D}_f(\mathbf{x}_0,\lambda)}{\epsilon_L} > m > (1+\epsilon_L)\mathbf{D}_f(\mathbf{x}_0,\lambda)$, *for some constant* ϵ_L,
- m *is sufficiently large,*
- $\lambda \in \mathcal{R}_{ON}$.

For any $\epsilon > 0$, *there exists a constant* $C = C(\epsilon, \epsilon_L)$ *and* $\sigma_0 = \sigma_0(\epsilon, \epsilon_L, n)$ *such that, whenever* $\sigma \leq \sigma_0$, *with probability* $1 - \exp(-C^2 \min\{m, \frac{m^2}{n}\})$, *we have,*

$$\left| \frac{\|\mathbf{x}^*_{\ell_2} - \mathbf{x}_0\|_2^2}{\|z\|_2^2} - \frac{\mathbf{D}_f(\mathbf{x}_0,\lambda)}{m - \mathbf{D}_f(\mathbf{x}_0,\lambda)} \right| < \epsilon$$

Optimal Regularizer Parameter:

$$\boxed{\lambda_{\text{best}} = \arg\min_{\lambda \geq 0} \mathbf{D}_f(\mathbf{x}_0,\lambda)}$$

Example

$\mathbf{X}_0 \in \mathbb{R}^{n\times n}$ is rank r. Observe, $\mathbf{y} = \mathcal{A}(\mathbf{X}_0) + \mathbf{z}$, solve the Matrix LASSO,

$$\min_{\mathbf{X}} \{\|\mathbf{y} - \mathcal{A}(\mathbf{X})\|_2 + \lambda\|\mathbf{X}\|_\star\}$$

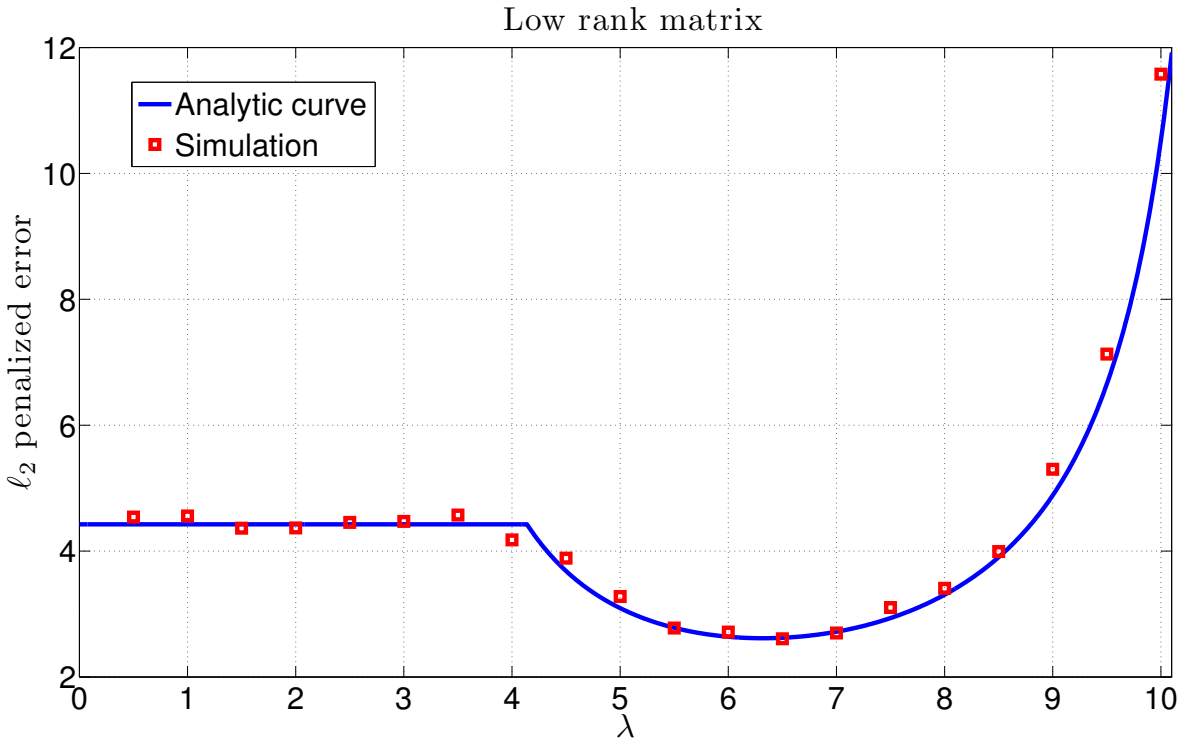


Figure : $n = 45$, $r = 6$, measurements $m = 0.6n^2$.

ℓ_2^2-LASSO: Connection to ℓ_2-LASSO

$$\mathbf{x}^*_{\ell_2} = \arg\min_{\mathbf{x}} \{\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 + \lambda f(\mathbf{x})\}$$

$$\boxed{\frac{\mathbf{A}^T(\mathbf{y} - \mathbf{A}\mathbf{x}^*_{\ell_2})}{\|\mathbf{y} - \mathbf{A}\mathbf{x}^*_{\ell_2}\|_2} \in \lambda \partial f(\mathbf{x}^*_{\ell_2})}$$

$$\mathbf{x}^*_{\ell_2^2} = \arg\min_{\mathbf{x}} \{\tfrac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \sigma\tau f(\mathbf{x}$$

$$\boxed{\mathbf{A}^T(\mathbf{y} - \mathbf{A}\mathbf{x}^*_{\ell_2^2}) \in \sigma\tau \partial f(\mathbf{x}^*_{\ell_2^2})}$$

Connection

Fix any $\lambda \geq 0$. Suppose $\mathbf{x}^*_{\ell_2}$ is optimal for ℓ_2-LASSO for that λ and $\|\mathbf{y} - \mathbf{A}\mathbf{x}^*_{\ell_2}\|_2 \neq 0$. Then, it is also optimal for the ℓ_2^2-LASSO for

$$\sigma\tau = \|\mathbf{y} - \mathbf{A}\mathbf{x}^*_{\ell_2}\|\lambda.$$

ℓ_2-ℓ_2^2 calibration

Formula

Define,

$$calib(\lambda) = \frac{m - \mathbf{D}_f(\mathbf{x}_0, \lambda) - \mathbf{C}_f(\mathbf{x}_0, \lambda)}{\sqrt{m - \mathbf{D}_f(\mathbf{x}_0, \lambda)}}, \qquad map(\lambda) = \lambda \cdot calib(\lambda)$$

Then, choosing,

$$\tau = map(\lambda)$$

connects ℓ_2 to ℓ_2^2 LASSO.

Intuition

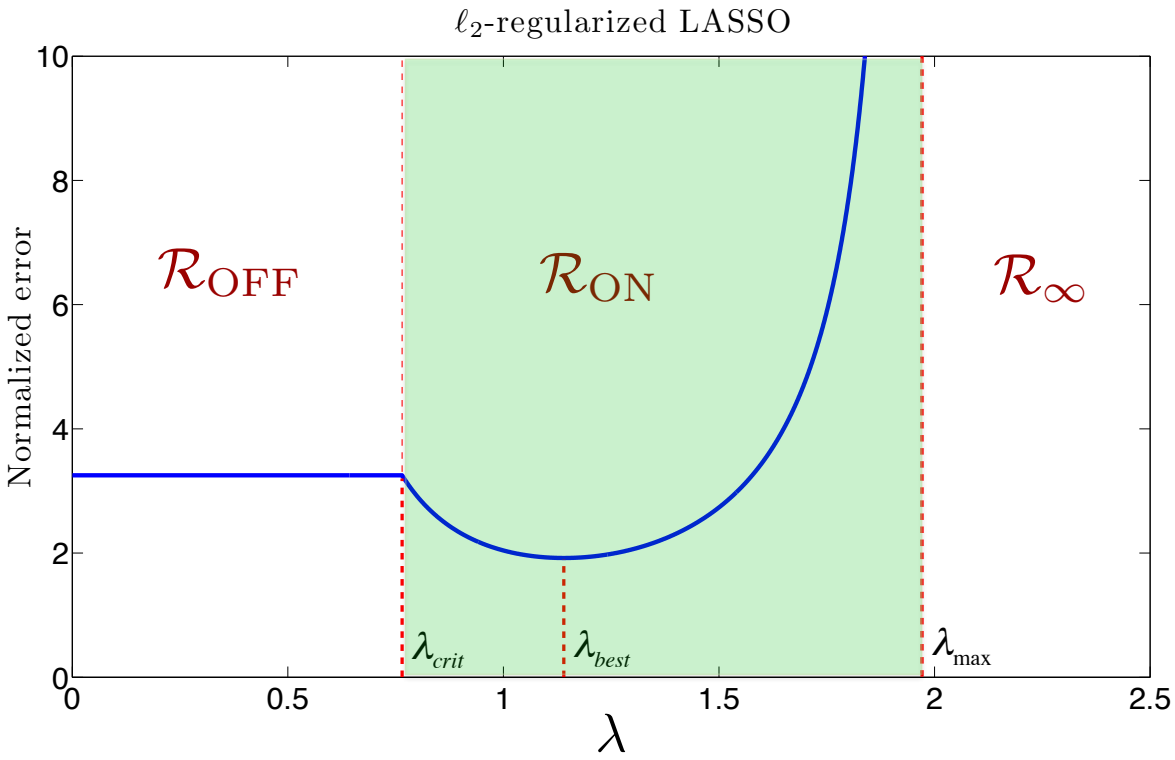
Recall: $\left\{ \|\mathbf{A}\mathbf{w}_{\ell_2}^* - \mathbf{z}\|_2 + \max_{\mathbf{s} \in \lambda \partial f(\mathbf{x}_0)} \mathbf{s}^T \mathbf{w}_{\ell_2}^* \right\} \approx \sigma \sqrt{m - \mathbf{D}_f(\mathbf{x}_0, \lambda)}$.

Conjecture: $\max_{\mathbf{s} \in \lambda \partial f(\mathbf{x}_0)} \mathbf{s}^T \mathbf{w}_{\ell_2}^* \approx \sigma \frac{\mathbf{C}_f(\mathbf{x}_0, \lambda)}{\sqrt{m - \mathbf{D}_f(\mathbf{x}_0, \lambda)}}$.

Proof based on same Framework?

Properties of the mapping

- map(λ) **takes** $\mathcal{R}_{\mathbf{ON}}$ **to** $\mathbb{R}^+$
  - map(λ) is *bijective* and *strictly increasing*.
  - map(λ_{crit}) $= 0$, map(λ_{max}) $= \infty$.

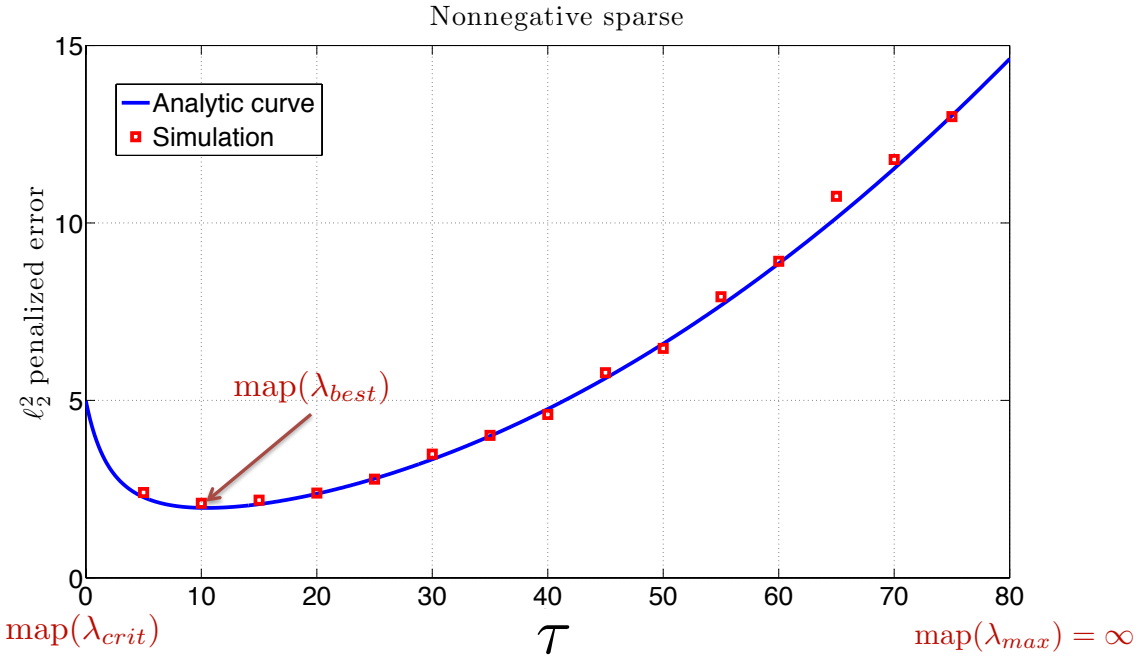


$\Longrightarrow$ R_{ON} is indeed the most important region of ℓ_2-LASSO.

Main Result on ℓ_2^2-LASSO

Formula (ℓ_2^2-LASSO error)

$$\frac{\|\mathbf{x}^*_{\ell_2^2} - \mathbf{x}_0\|}{\|z\|_2^2} \approx \frac{\mathbf{D}_f(\mathbf{x}_0, map^{-1}(\tau))}{m - \mathbf{D}_f(\mathbf{x}_0, map^{-1}(\tau))}$$



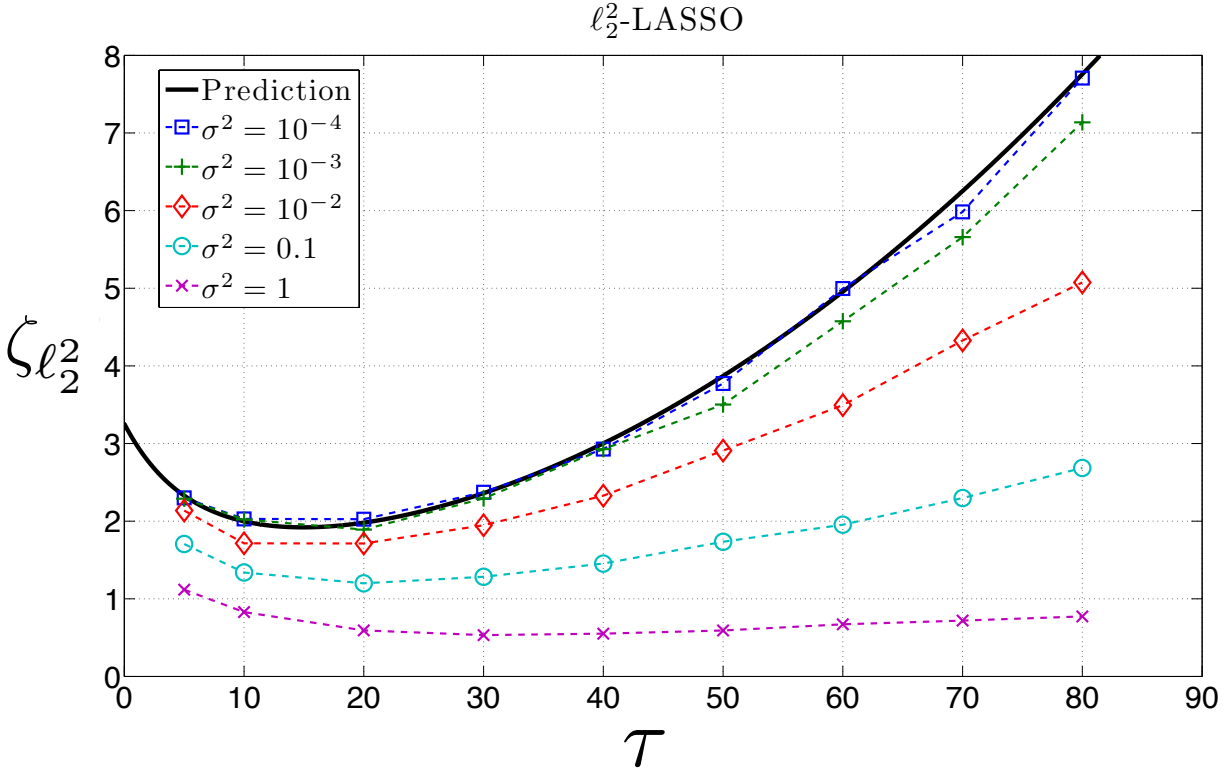
$\frac{k}{n} = 0.1$. $\frac{m}{n} = 0.3$, $n = 1000$, $\sigma^2 = 10^{-5}$.

Long Story Short

	Normalized Squared Error	Optimal Tuning
C-LASSO	$\frac{\mathbf{\Delta}_f(\mathbf{x}_0)}{m-\mathbf{\Delta}_f(\mathbf{x}_0)}$	
ℓ_2-LASSO	$\frac{\mathbf{D}_f(\mathbf{x}_0,\lambda)}{m-\mathbf{D}_f(\mathbf{x}_0,\lambda)}$ for $\lambda \in \mathcal{R}_{\text{ON}}$	$\lambda^* = \arg\min \mathbf{D}_f(\mathbf{x}_0, \lambda)$
ℓ_2^2-LASSO	$\frac{\mathbf{D}_f(\mathbf{x}_0,\text{map}^{-1}(\tau))}{m-\mathbf{D}_f(\mathbf{x}_0,\text{map}^{-1}(\tau))}$ for $\tau \in \mathbb{R}^+$	$\tau^* = \lambda^* \sqrt{m - \mathbf{D}_f(\mathbf{x}_0, \lambda^*)}$

- Our expressions are precise for sufficiently small noise level σ.
- They are upper bounds for arbitrary values of σ. Worst case error happens as $\sigma \to 0$.
- Converse results: When $m < \mathbf{\Delta}_f(\mathbf{x}_0)$, *not* robust recovery.
- Similar formulae for z arbitrary and z worst-case.

Example: Combining ℓ_2^2 and variance predictions



$n = 1000,\ \frac{k}{n} = 0.1,\ \frac{m}{n} = 0.5$

Summary and Conclusion

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- All results in noiseless structured signal recovery (compressed sensing, etc.) follow as special cases