Analysis of Compressive Sensing in Radar

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Compressed Sensing and Its Applications
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Overview

Several radar setups with compressive sensing approaches

- Range-Doppler resolution via compressive sensing
- Sparse MIMO Radar
- Antenna arrays with randomly positioned antennas
Time-Frequency Structured Random Matrices

Resolution of Range-Doppler in Radar
Resolution of Range-Doppler

Received signal is superposition of delayed and modulated (Doppler shifted) versions of sent signal.

Task: Determine delays (corresponding to distances; range) and Doppler shifts (corresponding to radial speed) from subsampled receive signal!
Gabor Systems in Finite Dimensions

Translation and Modulation on $\mathbb{C}^m$

$$(T^k g)_j = g(j-k) \mod m \quad \text{and} \quad (M^\ell g)_j = e^{2\pi i \ell j/m} g_j.$$ 

Time-frequency shifts

$$\pi(\lambda) = M^\ell T^k, \quad \lambda = (k, \ell) \in \{0, \ldots, m-1\}^2.$$ 

For $g \in \mathbb{C}^m$ define Gabor synthesis matrix ($\omega = e^{2\pi i / m}$)

$$\Psi_g = (\pi(\lambda) g)_{\lambda \in \{0, \ldots, m-1\}^2} = \begin{pmatrix} g_0 & g_{m-1} & \cdots & g_1 \\ g_1 & g_0 & \cdots & g_2 \\ g_2 & g_1 & \cdots & g_3 \\ g_3 & g_2 & \cdots & g_4 \\ \vdots & \vdots & \ddots & \vdots \\ g_{m-1} & g_{m-2} & \cdots & g_0 \\ \end{pmatrix} = \begin{pmatrix} g_0 & \cdots & g_1 & \cdots & \omega^{m-1} g_1 \\ \omega g_1 & \cdots & \omega g_2 & \cdots & \omega^{m-1} g_2 \\ \omega^2 g_2 & \cdots & \omega^2 g_3 & \cdots & \omega^{2(m-1)} g_3 \\ \omega^3 g_3 & \cdots & \omega^3 g_4 & \cdots & \omega^{3(m-1)} g_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega^{m-1} g_{m-1} & \cdots & \omega^{m-1} g_0 & \cdots & \omega^{(m-1)^2} g_0 \\ \end{pmatrix}. $$

Use of $\Psi_g \in \mathbb{C}^{m \times m^2}$ as measurement matrix in compressive sensing
Gabor Systems in Finite Dimensions

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\[
\Psi_g = (\pi(\lambda)g)_{\lambda \in \{0, \ldots, m-1\}^2}
\]

\[
= \begin{pmatrix}
g_0 & g_1 & \cdots & g_{m-1} \\
g_1 & g_0 & \cdots & g_2 \\
g_2 & g_1 & \cdots & g_3 \\
g_3 & g_2 & \cdots & g_4 \\
\vdots & \vdots & \ddots & \vdots \\
g_{m-1} & g_{m-2} & \cdots & g_0
\end{pmatrix}
\begin{pmatrix}
g_0 & \cdots & g_1 \\
\omega g_1 & \cdots & \omega g_2 \\
\omega^2 g_2 & \cdots & \omega^2 g_3 \\
\omega^3 g_3 & \cdots & \omega^3 g_4 \\
\vdots & \vdots & \ddots & \vdots \\
\omega^{m-1} g_{m-1} & \cdots & \omega^{m-1} g_0
\end{pmatrix}
\begin{pmatrix}
g_1 \\
\omega g_2 \\
\omega^2 g_3 \\
\omega^3 g_4 \\
\vdots \\
\omega^{m-1} g_{m-1}
\end{pmatrix}.
\]

Use of $\Psi_g \in \mathbb{C}^{m \times m^2}$ as measurement matrix in compressive sensing
Radar model (Herman, Strohmer 2008)

Emitted signal: \( g \in \mathbb{C}^m \).

Objects scatters \( g \) and radar device receives the contribution

\[
x_{\lambda} \pi(\lambda) g = x_{k,\ell} M^\ell T^k g.
\]

\( T^k \) corresponds to delay, i.e., distance of object
\( M^\ell \) corresponds to Doppler shift, i.e., speed of the object
\( x_{k,\ell} \) reflectivity of object

Received signal is superposition of contribution of all scatteres:

\[
y = \sum_{\lambda \in \Lambda} x_{\lambda} \pi(\lambda) g = \Psi g x.
\]

Usually few scatterers so that \( x \in \mathbb{C}^{m^2} \) can be assumed sparse.
Radar model (Herman, Strohmer 2008)

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Usually few scatterers so that \( x \in \mathbb{C}^{m^2} \) can be assumed sparse.

We will choose \( g \) as random vector below.
Reconstruction via compressive sensing

Reconstruction of $x$ from $y = Ax$ via $\ell_1$-minimization

\[
\begin{align*}
\min & \quad \|z\|_1 \\
\text{subject to} & \quad Az = y
\end{align*}
\]

\[
\begin{align*}
\min & \quad \|z\|_1 \\
\text{subject to} & \quad \|Az - y\|_2 \leq \eta
\end{align*}
\]
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\min \|z\|_1 \quad & \text{subject to } Az = y \\
\min \|z\|_1 \quad & \text{subject to } \|Az - y\|_2 \leq \eta
\end{align*}
\]

Alternatives:
Matching Pursuits
Iterative hard thresholding (pursuit)
Iteratively reweighted least squares
...

Uniform vs. nonuniform recovery

Often recovery results are for random matrices $A \in \mathbb{R}^{m \times N}$; choose generator $g \in \mathbb{C}^m$ for $\Psi_g$ at random

- **Uniform recovery**
  
  With high probability on $A$ every sparse vector is recovered;
  
  $$\mathbb{P}(\forall s\text{-sparse } x, \text{ recovery of } x \text{ is successful using } A) \geq 1 - \varepsilon.$$
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**Recovery conditions on \( A \)**

- Null space property
- Restricted isometry property
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**Recovery conditions on $A$**
- Null space property
- Restricted isometry property

- **Nonuniform recovery**
  A fixed sparse vector is recovered with high probability using $A \in \mathbb{R}^{m \times N}$;
  \[ \forall \text{s-sparse } x : \mathbb{P}(\text{recovery of } x \text{ is successful using } A) \geq 1 - \varepsilon. \]
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  **Recovery conditions on $A$**
  - Tangent cone (descent cone) of norm at $x$ intersects $\ker A$ trivially.
  - Dual certificates
Restricted isometry property (RIP)

Definition
The restricted isometry constant $\delta_s$ of a matrix $A \in \mathbb{C}^{m \times N}$ is defined as the smallest $\delta_s$ such that

$$(1 - \delta_s) \| x \|_2^2 \leq \| Ax \|_2^2 \leq (1 + \delta_s) \| x \|_2^2$$

for all $s$-sparse $x \in \mathbb{C}^N$. 

Stable and robust recovery

Theorem (Candès, Romberg, Tao ’04 – Cai, Zhang ’13)

Let $A \in \mathbb{C}^{m \times N}$ with $\delta_{2s} < 1/\sqrt{2} \approx 0.7071$. Let $x \in \mathbb{C}^N$, and assume that noisy data are observed, $y = Ax + e$ with $\|e\|_2 \leq \tau$. Let $x^\#$ by a solution of

$$\min_{z} \|z\|_1 \quad \text{such that} \quad \|Az - y\|_2 \leq \tau.$$ 

Then

$$\|x - x^\#\|_2 \leq C \frac{\sigma_s(x)_1}{\sqrt{s}} + D\tau,$$

$$\|x - x^\#\|_1 \leq C\sigma_s(x)_1 + D\sqrt{s}\tau$$

for constants $C, D > 0$, that depend only on $\delta_{2s}$. Here

$$\sigma_s(x)_1 = \inf_{z: \|z\|_0 \leq s} \|x - z\|_1.$$ 

Implies exact recovery in the $s$-sparse and noiseless case.
Dual certificate

**Theorem (Fuchs 2004, Tropp 2005)**

For $A \in \mathbb{C}^{m \times N}$, $x \in \mathbb{C}^N$ with support $S$ is the unique solution of

$$
\min \|z\|_1 \quad \text{subject to } Az = Ax
$$

if $A_S$ is injective and there exists a dual vector $h \in \mathbb{C}^{m}$ such that

$$(A^*h)_j = \text{sgn}(x_j), \quad j \in S, \quad |(A^*h)_\ell| < 1, \quad \ell \in \bar{S}.$$

**Corollary**

Let $a_1, \ldots, a_N$ be the columns of $A \in \mathbb{C}^{m \times N}$. For $x \in \mathbb{C}^N$ with support $S$, if the matrix $A_S$ is injective and if

$$
|\langle A_S^\dagger a_\ell, \text{sgn}(x_S) \rangle| < 1 \quad \text{for all } \ell \in \bar{S},
$$

then the vector $x$ is the unique $\ell_1$-minimizer with $y = Ax$.

Here, $A_S^\dagger$ is Moore-Penrose pseudo inverse.
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Corollary

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$$|\langle A^\dagger_S a_\ell, \text{sgn}(x_S) \rangle| < 1 \quad \text{for all } \ell \in \bar{S},$$

then the vector $x$ is the unique $\ell_1$-minimizer with $y = Ax$.

Here, $A^\dagger_S$ is Moore-Penrose pseudo inverse. One ingredient: Check that $\|A_S^* A_S - I\|_{2 \rightarrow 2} \leq \delta < 1.$
Stability and robustness via dual certificate

Theorem

Let \( \mathbf{x} \in \mathbb{C}^N \) and \( \mathbf{A} \in \mathbb{C}^{m \times N} \) with \( \ell_2 \)-normalized columns. Denote by \( S \subset [N] \) the indices of the \( s \) largest absolute entries of \( \mathbf{x} \). Assume that

(i) there is a dual certificate \( \mathbf{u} = \mathbf{A}^* \mathbf{h} \in \mathbb{C}^N \) with \( \mathbf{h} \in \mathbb{C}^m \) s.t.

\[
\mathbf{u}^\top = \text{sgn}(\mathbf{x})^\top, \quad \|\mathbf{u}^\top_c\|_\infty \leq \frac{1}{2}, \quad \|\mathbf{h}\|_2 \leq 3\sqrt{s}.
\]

(ii) \( \|\mathbf{A}^*\mathbf{A}^\top - I\|_2 \leq \frac{1}{2} \).

Given noisy measurements \( \mathbf{y} \) = \( \mathbf{A}\mathbf{x} + \mathbf{e} \in \mathbb{C}^m \) with \( \|\mathbf{e}\|_2 \leq \tau \), the solution \( \hat{\mathbf{x}} \) \( \in \mathbb{C}^N \) of noise-constrained \( \ell_1 \)-minimization satisfies

\[
\|\mathbf{x} - \hat{\mathbf{x}}\|_2 \leq 52\sqrt{s}\tau + 16\sigma_s(\mathbf{x})_1.
\]
Stability and robustness via dual certificate

Theorem

Let \( \mathbf{x} \in \mathbb{C}^N \) and \( \mathbf{A} \in \mathbb{C}^{m\times N} \) with \( \ell_2 \)-normalized columns. Denote by \( S \subset [N] \) the indices of the \( s \) largest absolute entries of \( \mathbf{x} \). Assume that

(i) there is a dual certificate \( \mathbf{u} = \mathbf{A}^* \mathbf{h} \in \mathbb{C}^N \) with \( \mathbf{h} \in \mathbb{C}^m \) s.t.

\[ \mathbf{u}^T = \text{sgn}(\mathbf{x})^T, \quad \|\mathbf{u}^T\|_\infty \leq \frac{1}{2}, \quad \|\mathbf{h}\|_2 \leq 3\sqrt{s}. \]

(ii) \( \|\mathbf{A}^* \mathbf{A}^T - \mathbf{I}\|_2 \leq \frac{1}{2} \).

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\[ \|\mathbf{x} - \hat{\mathbf{x}}\|_2 \leq 52\sqrt{s}\tau + 16\sigma_s(\mathbf{x})_1. \]

Remark: Error bound is worse by factor of \( \sqrt{s} \) than the one obtained from RIP. Can be removed again by additionally requiring the weak RIP.
Random choice of generator $g$

Recall Gabor synthesis matrix

$$\psi_g = (M^\ell T^k g)_{(k,\ell) \in [m]^2} \in \mathbb{C}^{m \times m^2}$$
Random choice of generator $g$

Recall Gabor synthesis matrix

$$\psi_g = (M^\ell T^k g)_{(k,\ell) \in [m]^2} \in \mathbb{C}^{m \times m^2}$$

Choice of $g$ as subgaussian random vector:
Entries of $g$ are independent, mean-zero, variance one and subgaussian: $\mathbb{P}(|g_j| \geq t) \leq 2e^{-Kt^2}$ for some $K > 0$.
Examples:

- Rademacher: entries $\pm 1$ with equal probability
- Steinhaus: entries are uniformly distributed on complex torus $\{z \in \mathbb{C} : |z| = 1\}$
- Gaussian: entries are standard real or complex Gaussian variables
RIP estimate for random generator (Krahmer, Mendelson, Rauhut 2014)

Theorem

Let $\Psi_g \in \mathbb{C}^{m \times N}$, $N = m^2$, be generated by a subgaussian random vector $g$. If, for $\delta \in (0, 1)$,

$$m \geq C \delta^{-2} s \max\{\log^2 s \log^2 N, \log(\varepsilon^{-1})\},$$

then with probability at least $1 - \varepsilon$ the restricted isometry constant of $\frac{1}{\sqrt{m}} \Psi_g$ satisfies $\delta_s \leq \delta$.

Implies stable and robust recovery via $\ell_1$ minimization with high probability if $m \geq Cs \log^2(s) \log^2(N)$.
RIP estimate for random generator (Krahmer, Mendelson, Rauhut 2014)

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Previous results:
Pfander, R, Tropp 2012: $m \geq C_\delta s^{3/2} \log^3 N$

Nonuniform recovery, Pfander, R 2010: $m \geq Cs \log(N)$

Theorem can be generalized to certain other systems of operators (instead of time-frequency shifts).
Numerical experiments for Steinhaus $g$

Horizontal axis $1/m = m/m^2$, vertical axis $s/m$.
Contours of success probability, 93% success rate, $1/(2 \log(m))$.
Numerical experiments suggest $s \leq \frac{m}{2 \log(m)}$ ensures $s$-sparse recovery.
Proof ingredient: chaos processes

Recall: $\delta_s$ is smallest constant such that

$$(1 - \delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_s)\|x\|_2^2$$

Equivalently, with $T_s = \{ x \in \mathbb{C}^N : \|x\|_2 \leq 1, \|x\|_0 \leq s \}$

$$\delta_s = \sup_{x \in T_s} |\|Ax\|_2^2 - \|x\|_2^2|$$
Proof ingredient: chaos processes

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$$\delta_s = \sup_{x \in T_s} \|\|Ax\|_2^2 - \|x\|_2^2\|$$

In our case

$$Ax = \frac{1}{\sqrt{m}} \Psi_g x = \frac{1}{\sqrt{m}} \sum_{k,\ell=0}^{m-1} x_{k,\ell} M^\ell T^k g = V_x g,$$

with $V_x = \frac{1}{\sqrt{m}} \sum_{k,\ell=0}^{m-1} x_{k,\ell} M^\ell T^k$. Since entries of $g$ have mean zero and variance one, $E\|V_x g\|_2^2 = \|x\|_2^2$, so that

$$\delta_s = \sup_{x \in T_s} \|\|V_x g\|_2 - E\|V_x g\|_2\|$$

This is a second order chaos processes.
Generic Chaining for Chaos Processes

Theorem (Krahmer, Mendelson, R 2014)

Let \( \mathcal{B} = -\mathcal{B} \subset \mathbb{C}^{m \times N} \) be a symmetric set of matrices and \( \xi \in \mathbb{C}^N \) be a subgaussian random vector. Then

\[
\mathbb{E} \sup_{\mathcal{B} \in \mathcal{B}} \left| \| B \xi \|_2^2 - \mathbb{E} \| B \xi \|_2^2 \right| 
\leq C_1 \gamma_2(\mathcal{B}, \| \cdot \|_{2\to2})^2 + C_2 \Delta_{\| \cdot \|_F}(\mathcal{B}) \gamma_2(\mathcal{B}, \| \cdot \|_{2\to2}).
\]

Here, \( \| B \|_F = \sqrt{\text{tr}(B^*B)} \) denotes the Frobenius norm.

Symmetry assumption \( \mathcal{B} = -\mathcal{B} \) can be dropped at the cost of slightly more complicated bound.

Here, \( \Delta_{\| \cdot \|}(\mathcal{B}) \) is the diameter of \( \mathcal{B} \) with respect to \( \| \cdot \| \) and \( \gamma_2(\mathcal{B}, \| \cdot \|) \) is Talagrand’s \( \gamma_2 \)-functional which can be bounded by

\[
\gamma_2(\mathcal{B}, \| \cdot \|) \leq C \int_0^{\Delta_{\| \cdot \|}(\mathcal{B})} \sqrt{\log N(\mathcal{B}, \| \cdot \|, u)} \, du,
\]

where \( N(\mathcal{B}, \| \cdot \|, u) \) are the covering numbers of \( \mathcal{B} \) at radius \( u \).
Tail bound

Theorem (Krahmer, Mendelson, R ’14 – Dirksen ’15)

Let $B = -B \subset \mathbb{C}^{m \times N}$ and $\xi \in \mathbb{C}^N$ be a subgaussian random vector. Then

$$
P \left( \sup_{B \in \mathcal{B}} \left\| B \xi \right\|_2^2 - \mathbb{E} \left\| B \xi \right\|_2^2 \geq C_1 E + t \right) \leq 2 \exp \left( -C_2 \min \left\{ \frac{t^2}{V^2}, \frac{t}{U} \right\} \right),$$

where

$$E := \Delta_{\| \cdot \|_F (\mathcal{B})} \gamma_2 (\mathcal{B}, \| \cdot \|_2 \rightarrow_2) + \gamma_2 (\mathcal{B}, \| \cdot \|_2 \rightarrow_2)^2,$$

$$V := \Delta_{\| \cdot \|_2 \rightarrow_2} \Delta_{\| \cdot \|_F (\mathcal{B})},$$

$$U := \Delta^2_{\| \cdot \|_2 \rightarrow_2} (\mathcal{B}).$$
Sparse MIMO radar
MIMO Radar
MIMO Radar in 2D

- $N_T$ transmit antennas at locations

  $$(0, (k - 1)d_T \lambda), \quad k = 1, 2, \ldots, N_T$$

- $N_R$ receive antennas at locations

  $$(0, (j - 1)d_R \lambda), \quad j = 1, \ldots, N_R$$

Choose $d_T = \frac{1}{2}$, $d_R = \frac{N_T}{2}$.

Then system has similar characteristics as antenna array with $N_T N_R$ antennas.

(Alternatively, $d_T = \frac{N_R}{2}$, $d_R = \frac{1}{2}$)
Measurement model
Strohmer, Friedlander 2012; Yu, Petropulu, Poor, 2011

- Transmit antennas send periodic continuous-time complex Gaussian pulses $s_1, \ldots, s_{N_T}$ with period $T$ and band-width $B$. 
Measurement model
Strohmer, Friedlander 2012; Yu, Petropulu, Poor, 2011

- Transmit antennas send periodic continuous-time complex Gaussian pulses $s_1, \ldots, s_{NT}$ with period $T$ and band-width $B$.

- Echo of target of unit reflectivity at position $(r \cos(\theta), r \sin(\theta))$ and radial speed $v$ at receiver $j$:

\[
r_j(t) = \sum_{k=1}^{NT} e^{2\pi i c \lambda^{-1}(t-d_{k,j}(t)/c)} s_k(t - d_{k,j}(t)/c)
\]

with carrier frequency $\lambda$, speed of light $c$, and distance from $k$th transmitter to target and from target to $j$th receiver

\[
d_{k,j}(t) = 2(r + vt) + \sin(\theta)d_T(k - 1)\lambda + \sin(\theta)(j - 1)d_R\lambda
\]

- Demodulation (multiplication of $r_j(t)$ with $e^{-2\pi i c \lambda^{-1}t}$) and assuming $B \ll \lambda$ (narrowband transmit waveforms), $v \ll c$ (slowly moving targets), $r \gg \lambda N_R N_T/2$ (far field scenario) yields measurements

\[
y_j(t) \approx e^{2\pi i \cdot 2\lambda^{-1}r} e^{2\pi i \sin(\theta)d_R(j-1)} \sum_{k=1}^{N} e^{2\pi i \cdot 2\lambda^{-1}vt} e^{2\pi i \sin(\theta)d_T(k-1)} s_k(t-2r/c)
\]
Discretization

- By the Shannon-Nyquist sampling theorem, the band-limited periodic complex Gaussian transmit signals can be represented by their sampled counterparts $s_k \in \mathbb{C}^{N_t}$ (sampled over one period $[0, T]$); $N_t$: number of samples.
Discretization

- By the Shannon-Nyquist sampling theorem, the band-limited periodic complex Gaussian transmit signals can be represented by their sampled counterparts \( s_k \in \mathbb{C}^{N_t} \) (sampled over one period [0, \( T \)]); \( N_t \): number of samples.
- Target is described by triple \((\theta, r, v)\) (azimuth, range, velocity); Discretization of \((\beta, \tau, f) = (\sin(\theta), 2r/c, 2\lambda^{-1}v)\) with stepsizes

\[
\Delta_{\beta} = \frac{2}{N_T N_R}, \quad \Delta_{\tau} = \frac{1}{2B}, \quad \Delta_f = \frac{1}{T}
\]

yields grid

\[ G = \{ (\beta \Delta_{\beta}, \tau \Delta_{\tau}, f \Delta_f) : \beta \in [N_R N_T], \tau \in [N_t], f \in [N_t] \} \]

Index set \( G = [N_T N_R] \times [N_t] \times [N_t] \) of size \( N := N_R N_T N_t^2 \) (here \([k] = \{1, \ldots, k\}\)).
Discretization grid

\[ N_R = N_T = 8 \]
Measurement Model I

One target with unit reflectivity at grid point indexed by
\( \Theta = (\beta, \tau, f) \in G \)

Discrete time samples at receiver \( j \) (\( j = 1, \ldots, N_R \))

\[ y_j = (y_j(\Delta_t), y_j(2\Delta_t), \ldots, y_j(N_t\Delta_t))^T \]

\[ = e^{2\pi i \cdot c\lambda^{-1}\tau\Delta_t} \left[ e^{2\pi i \cdot d_R\beta\Delta_{\beta}(j-1)} \sum_{k=1}^{N_T} e^{2\pi i \cdot d_T\beta\Delta_{\beta}(k-1)} M_f T_{\tau} s_k \right] \in \mathbb{C}^{N_t} \]

with translation and modulation operators on \( \mathbb{C}^{N_t} \) defined as

\[ (T_\tau s)_k = s_{k-\tau}, \quad (M_f s)_k = e^{2\pi i \cdot \frac{f_k}{N_t}} (s)_k. \]

Targets on grid points index by \( \Theta \in G \) with reflectivities \( \rho_\Theta \), setting also \( x_\Theta = e^{2\pi i \cdot c\lambda^{-1}\tau\Delta_t} \rho_\Theta \); measurements at receiver \( j \):

\[ y_j = \sum_{\Theta \in G} x_\Theta e^{2\pi i \cdot d_R\beta\Delta_{\beta}(j-1)} \sum_{k=1}^{N_T} e^{2\pi i \cdot d_T\beta\Delta_{\beta}(k-1)} M_f T_{\tau} s_k \]

\[ =: A_j^\Theta \]
Measurement Model II

Collection of sampled signals at all receivers:

\[
y = \begin{pmatrix} y_1 \\ \vdots \\ y_{NR} \end{pmatrix} = \begin{pmatrix} \sum_{\Theta \in G} x_{\Theta} A_{\Theta}^1 \\ \vdots \\ \sum_{\Theta \in G} x_{\Theta} A_{\Theta}^{NR} \end{pmatrix} = Ax \in \mathbb{C}^{N_r \cdot N_t}
\]

Measurement matrix

\[
A = \begin{pmatrix} A_{\Theta}^1 \\ \vdots \\ A_{\Theta}^{NR} \end{pmatrix}_{\Theta \in G} \in \mathbb{C}^{N_R N_t \times N_R N_T N_t^2}, \quad G = [N_R N_T] \times [N_t] \times [N_t]
\]

\[
A^j_{\Theta} = e^{2\pi i \cdot d_R \beta \Delta \beta (j-1)} \sum_{k=1}^{N_T} e^{2\pi i \cdot d_T \beta \Delta \beta (k-1)} M_f T_T s_k \in \mathbb{C}^{N_t}, \quad \Theta = (\beta, \tau, f)
\]

Structured random matrix; the \(s_1, \ldots, s_{N_T}\) are independent subgaussian random vectors, e.g. standard complex Gaussian random vectors, Rademacher vectors, or Steinhaus vectors

Number of measurements: \(m = N_R N_t\), signal dimension \(N = N_R N_T N_t^2\), i.e., \(m \ll N\); recall \(d_T = 1/2\), \(d_R = N_T/2\), \(\Delta \beta = \frac{2}{N_T N_R}\)
Reconstruction via Compressive Sensing

Reconstruction problem of solving $Ax = y$ is underdetermined.
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In many situations only very few targets are present, i.e., the vector $x$ of reflectivities is sparse!
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Use compressive sensing for reconstruction!

For recovery, we will study $\ell_1$-minimization

$$\min_z \|z\|_1 \quad \text{subject to } \|Ax - y\|_2 \leq \tau$$

and LASSO

$$\min_z \frac{1}{2} \|Az - y\|_2^2 + \lambda \|z\|_1$$
Recovery for random support sets

Strohmer and Friedlander (2013) showed recovery of the correct support via (debiased) LASSO for $s$-sparse signals with \textit{random support} (and random signs) with high probability under the condition

$$m = N_R N_t \geq Cs \log(N)$$

(plus minor additional technical assumptions).

Proof is based on an analysis of the coherence of $A$ and a general recovery result for random signals due to Tropp (2008).
Strohmer and Friedlander (2013) showed recovery of the correct support via (debiased) LASSO for $s$-sparse signals with random support (and random signs) with high probability under the condition

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Question:
Can we avoid the assumption of randomness of the support?
The RIP for MIMO radar measurements

Theorem (Dorsch, R 2015)

If

\[ N_t \geq C \delta^{-2} s \max \{ \log^2(s) \log^2(N), \log(\varepsilon^{-1}) \} \]

then the rescaled random radar measurement matrix

\[ \frac{1}{\sqrt{N_R N_T N_t}} A \in \mathbb{C}^{N_R N_t \times N_R N_T N_t^2} \]

satisfies \( \delta_s \leq \delta \) with probability at least \( 1 - \varepsilon \).

Implies stable and robust sparse recovery via \( \ell_1 \)-minimization.
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Proof uses generic chaining estimates for suprema of chaos processes (Krahmer, Mendelson, R 2014).
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Compared to other random matrix constructions in compressed sensing (where \( m \asymp s \log(e N/s) \) ) the result requires more measurements because here \( m = N_t N_R \); i.e., we suffer an additional factor of \( N_R \).
Almost optimality of RIP estimate

**Theorem (Dorsch, R 2015)**

If a realization of the random MIMO radar measurement matrix \( \frac{1}{\sqrt{N_R N_T N_t^2}} A \) satisfies \( \delta_s \leq 0.7 \) for \( s \leq N_t^2 \), then necessarily

\[
N_t \geq Cs \log(eN_t^2/s).
\]

**Proof idea:**

Introduce \( S_\beta := \{ (\beta', \tau', f') \in G : \beta' = \beta \} \).

If \( x \) has support in \( S_\beta \) then one can write

\[
Ax = a_R(\beta) \otimes Bx_{S_\beta}
\]

for a vector \( a_R(\beta) \in \mathbb{C}^{N_R} \) with entries of magnitude 1 and a matrix \( B \in \mathbb{C}^{N_t \times N_t^2} \). Applying lower sparse recovery bounds for \( B \) yields the claim.
Towards nonuniform recovery

Recovery depends on the fine structure of the support set:

Equivalence class of angles, $\beta, \beta' \in [N_R N_T],$

$$\beta \sim \beta' : \beta' - \beta \equiv 0 \mod N_R$$

This definition is motivated by the fact that the columns of $A$ satisfy, for $\Theta = (\beta, \tau, f), \Theta' = (\beta', \tau', f'),$

$$\langle A_\Theta, A_{\Theta'} \rangle = N_R \hat{\delta}_{\beta, \beta'} = \begin{cases} N_R & \text{if } \beta \sim \beta', \\ 0 & \text{otherwise.} \end{cases}$$
Towards nonuniform recovery

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Intuitively, the more elements of the support \( S \) are such that the corresponding \( \beta' \)'s are contained in different equivalent classes the better the matrix \( A_S \) is conditioned.
Well-balanced support sets

For a support set $S \subset G = [N_R N_T] \times [N_t] \times [N_t]$ let

$$S_{[\beta]} := \{ \Theta' = (\beta', \tau', f') \in S : \beta' \sim \beta \}.$$

Definition

A support set $S \subset G$ is called $\eta$-balanced, if for all angle classes $[\beta]$, $|S_{[\beta]}| \leq \eta |S|$. The parameter $\eta$ ranges in $[1, N]$. A small value of $\eta$ means that the support $S$ is well-distributed over the angle classes, which is favorable for recovery.
Well-balanced support sets

For a support set $S \subset G = [N_R N_T] \times [N_t] \times [N_t]$ let

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Definition

A support set $S \subset G$ is called $\eta$-balanced, if for all angle classes $[\beta]$,

$$|S[\beta]| \leq \eta \frac{|S|}{N_R}.$$  

The parameter $\eta$ ranges in $[1, N_R]$.  
A small value of $\eta$ means that the support $S$ is well-distributed over the angle classes, which is favorable for recovery.
Nonuniform recovery I

Theorem

Let \( \mathbf{x} \in \mathbb{C}^N \) and \( S \subset G \) be an index set corresponding to \( s \) largest absolute entries in \( \mathbf{x} \). Assume \( S \) to be \( \eta \)-balanced and that the signs of the coefficients \( \mathbf{x}_S \) form a Steinhaus sequence. Assume measurements \( \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n} \in \mathbb{C}^{N_R N_t} \) are given, where the signals \( \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{N_T} \) generating the measurement matrix \( \mathbf{A} \) are independent subgaussian random vectors, and \( \|\mathbf{n}\|_2 \leq \tau \). If

\[
m = N_R N_t \gtrsim \eta s \log^3 (N/\varepsilon),
\]

then, with probability at least \( 1 - \varepsilon \), the solution \( \mathbf{x}^\# \) to constrained \( \ell_1 \)-minimization satisfies

\[
\|\mathbf{x}^\# - \mathbf{x}\|_2 \leq C_1 \sigma_s(\mathbf{x})_1 + C_2 \frac{\tau \sqrt{s}}{\sqrt{N_T N_R N_t}},
\]

where \( C_1 \) and \( C_2 \) are numerical constants.

Exact recovery for \( s \)-sparse scene \( \mathbf{x} \).
Nonuniform recovery for LASSO

Theorem (Dorsch, R 2015)

Let \( x \in \mathbb{C}^N, N = N_R N_T N_t^2 \), be a fixed \( s \)-sparse target scene with \( \eta \)-balanced support \( S \) such that the phases of the nonzero entries form a random Steinhaus sequence and such that

\[
\min_{\Theta \in S} > 8 \sigma \sqrt{\frac{2 \log(N)}{N_T N_R N_t}}.
\]

Draw \( A \) at random and let \( y = A + e \) be noisy measurements with random noise, \( e \sim \mathcal{CN}(0, \sigma^2) \). Assume that

\[
m = N_R N_t \geq C \eta s \log^3(N/\varepsilon).
\]

Then, with probability at least \( 1 - 7 \max\{\varepsilon, N^{-3}\} \), the solution \( x^\# \) of

\[
\min_z \frac{1}{2} \|Az - y\|_2^2 + \lambda \|z\|_1 \quad \text{with} \quad \lambda = 2\sigma \sqrt{\frac{2 \log(N)}{N_T N_R N_t}}
\]

satisfies \( \text{supp}(x) = \text{supp}(x^\#) \).
Remarks about nonuniform recovery

- The debiased LASSO estimator $\hat{x} -$ least squares on $\text{supp}(x^\#)$, after computing LASSO solution – satisfies
  \[
  \|x - \hat{x}\|_2 \leq 2\sigma \sqrt{2s \log(N)/(N_T N_R N_t)}.
  \]

- The randomness in the signs of the nonzero entries of $x$ can likely be removed.

- For optimal balancedness parameter $\eta = 1$, we obtain a (near-)optimal bound on the number of measurements: $m \geq Cs \log^3(N/\varepsilon)$.

- RIP-result covers the worst case where $\eta = N_R$.

- A random support set will be $\eta$-balanced for small $\eta$ with high probability, which explains the result of Strohmer and Friedlander.
Numerical experiments for Doppler-free scenario

Success rates for various values of $\eta$

red curve corresponds to randomly chosen support sets

$N_T = N_R = 8$ transmit and receive antennas, $N_t = 64$
time-domain samples, grid size $N = N_T N_R N_t = 4096$
$m = N_R N_t = 512$ measurements
Antenna arrays with random antenna positions
Radar setup

$n$ antenna elements on square $[0, B]^2$ in plane $z = 0$. Targets in the plane $z = z_0$ on grid of resolution cells $r_j \in [-L, L]^2 \times \{z_0\}$, $j = 1, \ldots, N$ with mesh size $h$.

$x \in \mathbb{C}^N$: vector of reflectivities in resolution cells $(r_j)_{j=1,\ldots,N}$. 
Sensing mechanism (Fannjiang, Strohmer, Yan 2010)

Antenna at position $a \in \mathbb{R}^3$ emits monochromatic wave (wavelength $\lambda$, wavenumber $\omega$) with amplitude at position $r \in \mathbb{R}^3$ given by Green’s function of Helmholtz equation

$$H(a, r) = \frac{\exp{(2\pi i \|r - a\|_2/\lambda)}}{4\pi \|r - a\|_2}.$$  

Approximation (valid for large $z_0$): $H(a, r) \approx \frac{\exp{i\omega z_0}}{4\pi z_0} G(a, r)$ with

$$G(a, r) = \exp \left( \frac{i\omega}{2z_0} (|r_1 - a_1|^2 + |r_2 - a_2|^2) \right)$$

Signal corresponding to emitting antenna $a_\ell$ and receive antenna $a_k$ (Born approximation)

$$y_{(k, \ell)} = \sum_{j=1}^{N} x_j G(a_\ell, r_j) G(r_j, a_k) = (Ax)_{(k, \ell)}, \quad k, \ell = 1, \ldots, n$$

$n^2$ measurements
Random scattering matrix

Choose antenna positions $a_j$, $j \in [n]$, independently and uniformly at random in $[0, B]^2$. Then $A \in \mathbb{C}^{n^2 \times N}$ is structured random matrix. Entries

$$A_{(k, \ell):j} = G(a_k, r_j) G(r_j, a_\ell), \quad (k, \ell) \in [n]^2, j \in [N].$$

Define $v(a_k, a_\ell) = (G(a_k, r_j) G(r_j, a_\ell))_{j \in [N]} \in \mathbb{C}^N$. Then

$$A = \begin{pmatrix} v(a_1, a_1) \\ v(a_1, a_2) \\ \vdots \\ v(a_2, a_1) \\ \vdots \\ v(a_n, a_n) \end{pmatrix}$$

Rows and columns are coupled. Under the condition $\frac{hB}{\lambda z_0} \in \mathbb{N}$ we have $\mathbb{E} A^* A = I$.
Reconstruction via $\ell_1$-minimization

Sparse scene (sparsity $s = 100$, 6400 grid points):

Reconstruction ($n = 30$ antennas, 900 noisy measurements, SNR 20dB)
Nonuniform recovery

Theorem (Hügel, R, Strohmer 2014)

Let $\mathbf{x} \in \mathbb{C}^N$. Choose the $n$ antenna positions independent and uniformly at random in $[0, B]^2$. Assume $\frac{hB}{\lambda z_0} \in \mathbb{N}$, where $h$ is mesh size and $\lambda$ the wavelength; further

$$n^2 \geq C_s \ln^2(N/\varepsilon).$$

Let $\mathbf{y} = A\mathbf{x} + \mathbf{e} \in \mathbb{C}^{n^2}$ with $\|\mathbf{e}\|_2 \leq \eta n$. Let $\mathbf{x}^\#$ be the solution to

$$\min \|\mathbf{z}\|_1 \quad \text{subject to} \quad \|\mathbf{y} - A\mathbf{z}\|_2 \leq \eta n.$$

Then with probability at least $1 - \varepsilon$

$$\|\mathbf{x} - \mathbf{x}^\#\|_2 \leq C_1 \sigma_s(\mathbf{x})_1 + C_2 \sqrt{s} \eta.$$

Exact recovery when $\eta = 0$ and $\sigma_s(\mathbf{x})_1 = 0$. RIP estimate open.
Conclusions

Analysis of compressive sensing in various radar setups may be interesting and challenging!

- Time-Frequency (range-Doppler) structured random matrices (Pfander, R 2010; Pfander, R, Tropp 2012; Krahmer, Mendelson, R - 2014)
- MIMO radar with random transmit pulses (Friedlander, Strohmer 2014; Dorsch, R 2015)
- Antenna arrays with random antenna positions (Fannjiang, Strohmer 2013; Hügel, R, Strohmer 2014)

Not covered:

- Subsampled random convolutions (R, Romberg, Tropp 2012; Krahmer, Mendelson, R 2014)
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  ▶ ...
▶ More challenging mathematical problems from radar applications
  ▶ Off grid compressive sensing
  ▶ ...
The End

Literature