On existence of maximal semidefinite invariant subspaces

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We consider the class of J-dissipative operators in some Krein space E. Recall that a Krein space is a Hilbert space H endowed with the usual inner product (x, y) as well as an indefinite inner product $[x, y] = (Jx, y)$, where J is a self-adjoint operator such that $J^2 = 1$. A subspace $L \subset H$ is said to be nonnegative (positive, uniformly positive) if the inequality $[x, x] \geq 0$ ($[x, x] > 0$, $[x, x] \geq \delta_0 \|x\|^2$ (δ_0 is a positive constant)) holds for all $x \in L$. If a nonnegative subspace L admits no nontrivial nonpositive extensions then it is said to be maximal nonpositive. By analogy, we can define nonpositive, negative, and uniformly negative subspaces as well as maximal positive, uniformly positive subspaces, etc. Let A be a linear operator in H with domain $D(A)$. A densely defined operator A is said to be dissipative in H if $-\mathrm{Re}\,(Ax, x) \geq 0$ for all $x \in D(A)$. A dissipative operator is said to be maximal dissipative or m-dissipative if it admits no nontrivial dissipative extensions. An operator A is said to be J-dissipative (J-maximal dissipative) in the Krein space $(H, [\cdot, \cdot])$ if JA is a dissipative (m-dissipative) operator in H. Given a J-maximal dissipative operator $A : H \to H$, we discuss a problem of finding maximal nonnegative (nonpositive) subspaces invariant under the operator A. The problem on the existence of invariant maximal semidefinite subspaces turned out to be a focus of attention in the theory of operators in Pontryagin and Krein spaces. The first results were obtained in the celebrated article by Pontryagin M.S. in 1944. Later his results were generalized by many authors, in particular, by Krein M.G., Langer H., Jonas P., Dritschel M.A., Azizov T.Ya., and Shkalikov A.A. The latest results and the bibliography can be found in [1]-[3].

We refine some results on existence of invariant maximal semidefinite subspaces for J-dissipative operators and present also new necessary condition for the existence of these subspaces. The results are applied to the particular case when the operator A admits the representation in the form of an operator matrix in the canonical decomposition $H = H^+ + H^-$, ($H^\pm = P^\pm H$, $J = P^+ - P^-$, $P^\pm$ – othoprojections) and to some differential operators. The main conditions ensuring the existence of maximal semidefinite invariant subspaces are stated in the terms of the interpolation theory of Banach spaces (see, for instance, [4]). For example, assume that the imaginary axis belong to the resolvent set of A and put $H_1 = D(A)$ and H_{-1} is a completion of H in the norm $\|u\|_{-1} = \|A^{-1}u\|$ ($\|\cdot\|$ is the norm in H). Assume also that we have the estimate $(A - i\omega)^{-1}\| \leq c/(1+|\omega|)$ for all $\omega \in \mathbb{R}$ (c is a constant). In this case sufficient and necessary conditions of existence of semidefinite invariant subspaces are connected with the condition $(H_1, H_{-1})_{1/2,2} = H$.

References.

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