Non-negativity analysis for exponential-polynomial-trigonometric functions

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The class of exponential-polynomial-trigonometric (EPT) functions is ubiquitous in the mathematical sciences. It is the class of functions that appear as solutions to linear differential equations with constant coefficients. In linear dynamical systems theory they appear as impulse response functions and that context are often represented as $y(t) = ce^{At}b$, where c is a row vector, A a square matrix and b a column vector. The name EPT functions draws on the fact that these functions can be written in the form $\sum_{i=0}^{d} p_i(t)e^{\lambda_i t}\cos(\theta_i t)$, where the p_i are polynomials and λ_i and θ_i are real numbers. These functions also appear in probability theory and financial mathematics (e.g. as forward rate curves). In such applications non-negativity is often required. In the present paper we address the question of how to characterize non-negative EPT functions. We describe necessary conditions and we present a new sufficient condition and methods for verification of this sufficient condition. The main idea is to represent an EPT function as the product of a row vector $b(t)$ of EP functions and a column vector $F(t) := f(e^{i\theta_1 t}, \ldots, e^{i\theta_m t})$ of multivariate trigonometric polynomials with unimodular exponential functions $e^{i\theta_k t},\ k = 1, 2, \ldots, m$ as arguments; here $\theta_k,\ k = 1, 2, \ldots, m$ are chosen to be real numbers which are linearly independent over the set of rational numbers $\mathbf{Q}$. From the theory of almost periodic functions it follows that the closure of the set of vectors $F(t)$ obtained by varying t over the interval $[T, \infty)$ for any $T > 0$, is equal to the set $f(\mathbf{T}^m)$, where $\mathbf{T}^m$ denotes the $m-$dimensional unit torus in $\mathbf{C^m}$. This will be used to describe a sufficient condition. Two methods for verifying whether the sufficient condition is satisfied are presented, one based on minimization over the torus of a continuous level function which is non-negative iff the sufficient condition is satisfied; the other based on algebraic optimization techniques, involving Groebner basis

and Sturm chain calculations. The methods are based on the availability of a generalized Budan-Fourier sequence technique to determine the minimum of an EPT function on a given finite interval $[0, T]$ which is presented elsewhere by the authors.

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