Riemann boundary value problem on non-rectifiable arcs and Cauchy transform

Boris A. Kats

Let Γ be Jordan arc on the complex plane $\mathbb{C}$ with end points a_1 and a_2. We consider the Riemann boundary value problem on this arc, i.e., the problem on evaluation of holomorphic in $\overline{\mathbb{C}} \setminus \Gamma$ function $\Phi(z)$ satisfying equality

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), t \in \Gamma \setminus \{a_1, a_2\},$$

and certain restrictions on its behavior at the points a_1, a_2. Here $\Phi^\pm(t)$ are limit values of $\Phi(z)$ at a point $t \in \Gamma \setminus \{a_1, a_2\}$ from the left and from the right correspondingly, and functions G and g are given. In the simplest case $G \equiv 1$ the Riemann boundary value problem turns into so called jump problem.

This boundary value problem has a long history and a lot of applications, both traditional and new. The classic results on the problem are based on assumption, that the arc Γ is piecewise-smooth, or at least rectifiable. A solution of the jump problem under this assumption is Cauchy integral $\Phi(z) = (2\pi i)^{-1} \int_\Gamma g(t)(t-z)^{-1}dt$, and the Riemann boundary problem reduces to the jump problem by means of well-known factorization method.

In the present work we investigate this problem for non-rectifiable arcs. We introduce certain distributions with supports on non-rectifiable arc Γ, which generalize operation of weighted integration along this arc. Then we consider boundary behavior of Cauchy transforms of these distributions, i.e., their convolutions with $(2\pi i z)^{-1}$. As a result, we obtain description of solutions of the Riemann boundary value problem in terms of a new version of metric dimension of arc Γ, so called approximation dimension. It characterizes a rate of the best approximation of Γ by polygonal lines. For instance, if Γ is graph of a real function, then its approximation dimension is related with coefficients of Faber-Schauder expansion of this function.