Algorithms for Solving the Polynomial Eigenvalue Problem

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TU Berlin—March 13, 2006

Polynomial Eigenproblem

$$P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i, \qquad A_i \in \mathbb{C}^{n\times n}, \quad A_m \neq 0.$$

P assumed **regular** ($\det P(\lambda) \not\equiv 0$).

Find scalars λ and nonzero vectors x and y satisfying $P(\lambda)x = 0$ and $y^* P(\lambda) = 0$.

Hyperbolic and Overdamped Quadratics

$$Q(\lambda)x = (\lambda^2 A + \lambda B + C)x = 0.$$

Q is **hyperbolic** if A is Hermitian pos. def., B and C Hermitian, and

$$(x^* Bx)^2 > 4(x^* Ax)(x^* Cx) \quad \text{for all } x \neq 0.$$

Hyperbolic implies real e'vals with

$$\lambda_1 \geq \cdots \geq \lambda_n > \lambda_{n+1} \geq \cdots \geq \lambda_{2n}.$$

Q is **overdamped** if it is hyperbolic with B Hermitian pos. def. and C Hermitian pos. semidef.

Overdamped implies $\lambda_1 \leq 0$.

Methods

Interested in methods for solving dense problems.

- Solvent.
- Bandwidth reduction.
- Structure-preserving transformations.
- Linearization.

Solvent Approach

$Q(X) = AX^2 + BX + C.$
S is a **solvent** if $Q(S) = 0$. Then

$$Q(\lambda) = -(B + AS + \lambda A)(S - \lambda I).$$

Eigenproblem reduced to two $n \times n$ problems: standard and generalized.

Existence of solvent guaranteed if $|\lambda_1| \geq \cdots \geq |\lambda_n| > |\lambda_{n+1}| \geq \cdots \geq |\lambda_{2n}|$ and lin indep e'vecs exist for $\{ \lambda_1, \dots, \lambda_n \}$ and $\{ \lambda_{n+1}, \dots, \lambda_{2n} \}$.
Conditions satisfied for overdamped polys.

Solvent via Newton, Bernoulli

- Newton's method with exact line searches (H & Kim, 2001). Solve a gen Sylvester equation on each step.

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- Newton's method with exact line searches (H & Kim, 2001). Solve a gen Sylvester equation on each step.
- Bernoulli iteration (H & Kim, 2000):
  $(AX_i + B)X_{i-1} + C = 0, \quad X_1 = -A^{-1}B.$
  Convergence requires $|\lambda_n| > |\lambda_{n+1}|$ and existence of

  $$\begin{aligned} \text{dominant solvent: } & \Lambda(S_1) = \{ \lambda_1, \dots, \lambda_n \}, \\ \text{minimal solvent: } & \Lambda(S_2) = \{ \lambda_{n+1}, \dots, \lambda_{2n} \}. \end{aligned}$$

  Linear convergence to S_1 with constant $|\lambda_n|/|\lambda_{n+1}|$.
  H & Kim showed can be faster than `polyeig`.

Solvent via Cyclic Reduction

Cyclic reduction (Guo & Lancaster, 2005) on infinite block tridiagonal system with rows $[C\ B\ A]$.

- Matrix iteration $X_{i+1} = X_i - A_i - B_i^{-1} C_i$ etc. $6\frac{1}{3}n^3$ flops per iteration for **overdamped** Q.
- **Quadratic convergence** for **overdamped** Q.
- Dominant and minimal solvents are obtained from limit X as $-X^{-1}C$ and $-A^{-1}X$.
- E'vals **not guaranteed real**!
- Total cost if k iterations:

  $(25 + 6k)n^3$ flops e'vals only
  $(55 + 6k)n^3$ flops e'vals and e'vecs

Linearization for Hyperbolic Quadratic

Assume we know σ such that $Q(\sigma) < 0$.

$$\widetilde{Q}(\lambda) := Q(\lambda + \sigma) = \lambda^2 Q''(\sigma) + \lambda Q'(\sigma) + Q(\sigma)$$
$$\equiv \lambda^2 \underbrace{A}_{>0} + \lambda \widetilde{B} + \underbrace{\widetilde{C}}_{<0}.$$

$$X - \lambda Y = \begin{bmatrix} \widetilde{B} & A \\ A & 0 \end{bmatrix} - \lambda \begin{bmatrix} \widetilde{C} & 0 \\ 0 & -A \end{bmatrix}.$$

Transform $X - \lambda Y \to G - \lambda I$ using Cholesky of A and $\widetilde{C}$.
Real e'vals assured.
Total cost:

$18n^3$ flops e'vals only
$33n^3$ flops e'vals and e'vecs

Bandwidth Reduction

$Q(\lambda) = \lambda^2 A + \lambda B + C$.

Transform $G\,Q(\lambda)\,H = \lambda^2 \widetilde{A} + \lambda \widetilde{B} + \widetilde{C}$ with $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$ of minimal bandwidth in fte # operations. Then apply some other method.

- **Tridiagonal** form is not achievable ($2n^2$ parameters, $\sim 3n^2$ equations).
- Is **pentadiagonal** form achievable?

Structure-Preserving Transformations

Idea is to produce a diagonal poly with the same spectrum as P.

- Garvey, Prells & Friswell (2002, 2003)
- Chu & Del Buono (2005)

pursue the approach:

- Form a linearization (in $\mathbb{DL}(P)$).
- Iteratively transform it to diagonal form preserving its structure.
- Generate the transformations via isospectral flows?

Many open questions.

Linearizations

$$L(\lambda) = \lambda X + Y, \qquad X, Y \in \mathbb{C}^{mn\times mn}$$

is a **linearization** of $P(\lambda) = \sum_{i=0}^{m} \lambda^i A_i$ if

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(m-1)n} \end{bmatrix}$$

for some **unimodular** $E(\lambda)$ and $F(\lambda)$.

Example

Companion form linearization

$$E(\lambda)\left(\lambda \begin{bmatrix} A_2 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} A_1 & A_0 \\ -I & 0 \end{bmatrix}\right) F(\lambda) = \begin{bmatrix} \lambda^2 A_2 + \lambda A_1 + A_0 & 0 \\ 0 & I \end{bmatrix}.$$

Solution Process

Standard way of solving $P(\lambda)x = 0$:

- **Linearize** $P(\lambda)$ into $L(\lambda) = \lambda X + Y$.
- Solve **generalized eigenproblem** $L(\lambda)z = 0$.
- **Recover** eigenvectors of P from those of L.

Usual choice of L: companion linearization, for which

$$z = \begin{bmatrix} \lambda^{m-1}x \\ \vdots \\ \lambda x \\ x \end{bmatrix}.$$

Left e'vec: more complicated formula.

Desiderata for a Linearization

- Good conditioning.
- Backward stability.
- Preservation of structure, e.g. **symmetry**.
- Numerical preservation of key **qualitative** properties, including location and symmetries of spectrum.
- Preserve partial multiplicities of e'vals (strong linearization).

Meta-Algorithm

Preprocess *P*. E.g., Fan, Lin & Van Dooren (2004):

$$\lambda^2 A + \lambda B + C \to \mu^2(\gamma^2\delta A) + \mu(\gamma\delta B) + \delta C \qquad \begin{cases} \gamma = \sqrt{c/a} \\ \delta = 2/(c + b\gamma) \end{cases}$$

for one or more linearizations *L*

Balance *L*

Apply QZ or HZ to *L*

Obtain relevant e'vals.

Recover left and right e'vecs

Iteratively refine e'vecs

Compute/estimate b'errs and condition numbers

Detect nonregular problem

end

Balancing

Balancing GEP:

- Ward (1981)
- Lemonnier & Van Dooren (2005)

To investigate:

- Exploit structure of pencils arising via linearization of a matrix poly.
- Can we balance a QEP?
- To what extent balancing can make the results worse?
  Cf. Watkins (2005): *A Case where Balancing is Harmful*.

Iterative Refinement

- Underlying theory for fixed and extended precision residuals in Tisseur (2001).
- Done for definite GEPs in Davies, H & Tisseur (2001).
- Details for QEPs in Berhanu (2005), incl. complex conj. pairs in real arith.

Issues:

- Convergence to wrong eigenpair or non-convergence.
- Exploiting structure of pencil from a linearization.

$\mathbb{L}_1$ and $\mathbb{L}_2$ Linearizations

$$\Lambda := [\lambda^{m-1}, \lambda^{m-2}, \dots, 1]^T.$$

Mackey, Mackey, Mehl & Mehrmann (2005) define

$$\mathbb{L}_1(P) = \{\, L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\ v \in \mathbb{C}^m \,\},$$
$$\mathbb{L}_2(P) = \{\, L(\lambda) : (\Lambda^T \otimes I_n)L(\lambda) = w^T \otimes P(\lambda),\ w \in \mathbb{C}^m \,\}.$$

They show that

- $\mathbb{L}_1$ and $\mathbb{L}_2$ are **vector spaces** of dim $m(m-1)n^2 + m$.
- **Almost all** pencils in $\mathbb{L}_1$, $\mathbb{L}_2$ are linearizations of P.

Quadratic case ($m = 2$): $L = \lambda X + Y \in \mathbb{L}_1(P)$ iff

$$\begin{bmatrix} v_1 A_2 & v_1 A_1 & v_1 A_0 \\ v_2 A_2 & v_2 A_1 & v_2 A_0 \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} + Y_{11} & Y_{12} \\ X_{21} & X_{22} + Y_{21} & Y_{22} \end{bmatrix}.$$

$\mathbb{L}_1$ and $\mathbb{L}_2$ Linearizations cont.

Recall

$$\mathbb{L}_1(P) = \{\, L(\lambda) : L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\ v \in \mathbb{C}^m \,\}.$$

Note

$$L(\lambda)(\Lambda\otimes x) = L(\lambda)(\Lambda\otimes I_n)(1\otimes x) = (v\otimes P(\lambda))(1\otimes x) = v\otimes P(\lambda)x.$$

So (x, λ) **is an e'pair of** P **iff** $(\Lambda \otimes x, \lambda)$ **is an e'pair of** L**.**

- **Right** eigenvectors of P can be recovered from **right** eigenvectors of linearizations in $\mathbb{L}_1$.
- **Left** eigenvectors of P can be recovered from **left** eigenvectors of linearizations in $\mathbb{L}_2$.

$\mathbb{DL}(P)$ Linearizations

Mackey, Mackey, Mehl & Mehrmann (2005) define

$$\mathbb{DL}(P) = \mathbb{L}_1(P) \cap \mathbb{L}_2(P).$$

They show that

- $L \in \mathbb{DL}(P)$ iff $w = v$ in the definitions of $\mathbb{L}_1$ and $\mathbb{L}_2$.
- $\mathbb{DL}(P)$ is a vector space of dimension m.
- Almost all pencils in $\mathbb{DL}(P)$ are linearizations of P.

Example: For $Q(\lambda) = \lambda^2 A + \lambda B + C$, $\mathbb{DL}(Q)$ is the pencils

$$L(\lambda) = \lambda \begin{bmatrix} v_1 A & v_2 A \\ v_2 A & v_2 B - v_1 C \end{bmatrix} + \begin{bmatrix} v_1 B - v_2 A & v_1 C \\ v_1 C & v_2 C \end{bmatrix}, \ v \in \mathbb{C}^2.$$

Conditioning in $\mathbb{DL}(P)$

Let

$$\rho = \frac{\max_i \|A_i\|_2}{\min(\|A_0\|_2, \|A_m\|_2)} \geq 1.$$

H, D. S. Mackey & Tisseur (2005) show that

$$\kappa_L(\lambda; e_1) \leq \rho^2 m^{7/2} \kappa_P(\lambda), \quad A_0 \text{ nonsing}, \quad |\lambda| \geq 1,$$
$$\kappa_L(\lambda; e_m) \leq \rho^2 m^{7/2} \kappa_P(\lambda), \quad A_m \text{ nonsing}, \quad |\lambda| \leq 1.$$

- For Q not heavily damped, $\rho = O(1)$.
- With FLV scaling, $\rho = O(1)$ for **elliptic** quadratics.

Conditioning of Companion Form

H, D. S. Mackey & Tisseur (2005) find that κ_{C_1}/κ_P depends on

- ratios $\|w\|_2/\|y\|_2 \geq 1$ of norms of left e'vecs of C_1 and P,
- rational functions of the $\|A_i\|_2$ and λ.

Conclude that

- If $\|A_i\| \approx 1$, $i = 0\colon m$ then $\kappa_{C_1} \approx \kappa_P$.
- If $\|w\|_2/\|y\|_2 \gg 1$ or if $\|A_i\|_2 \ll 1$, $i = 0\colon m$, then $\kappa_{C_1} \gg \inf_v \kappa_L(\lambda; v)$ is possible.

Companion versus $\mathbb{DL}(P)$

- Companion is always a linearization; for $\mathbb{DL}(P)$ need spectrum of P distinct from "roots of v".
- $\mathbb{DL}(P)$ pencil symmetric if P is, companion not.
- Scaling can help both.
- Easier to check suff. conds for $\mathbb{DL}(P)$ well conditioned.

Role of $\mathbb{L}_1$ and $\mathbb{L}_2$

- Can preserve other structures of P.
- Conditioning analysis can be extended using new left e'vec recovery formula.

Concluding Remarks

- Only general-purpose current methods are those based on linearization.
- Recent work opens up opportunities for developing more sophisticated algs based on linearization.
- Interesting possibilities for hyperbolic polys.
- How to design an LAPACK QEP solver?

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