

Automated Multi-Level Substructuring for nonlinear eigenproblems

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- AMLS for linear eigenvalue problems
- AMLS for nonlinear eigenvalue problems
 - Gyroscopic problems
 - Nonproportional damping
 - Vibration of fluid–solid structure
- Conclusions

Automated Multi-Level Substructuring

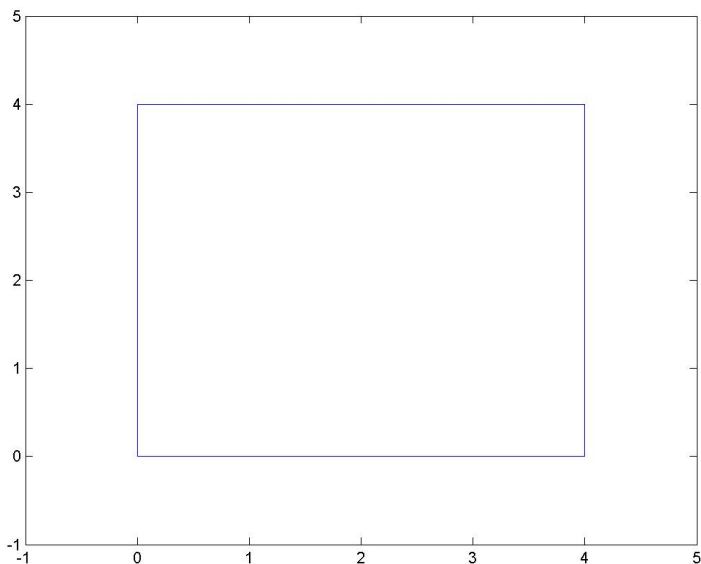
AMLS was introduced by [Bennighof \(1998\)](#) and was applied to huge problems of frequency response analysis.

The large finite element model is recursively divided into very many substructures on several levels based on the sparsity structure of the system matrices.

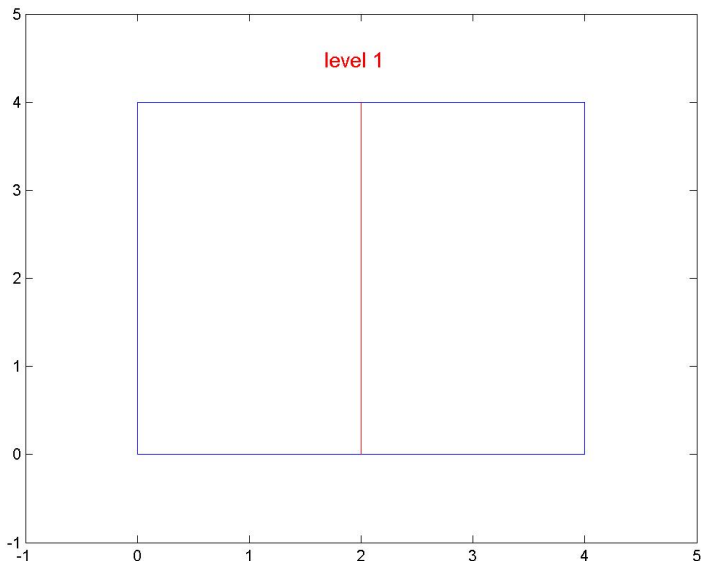
Assuming that the interior degrees of freedom of substructures depend quasistatically on the interface degrees of freedom, and modeling the deviation from quasistatic dependence in terms of a small number of selected substructure eigenmodes the size of the finite element model is reduced substantially yet yielding satisfactory accuracy over a wide frequency range of interest.

Recent studies in vibro-acoustic analysis of passenger car bodies where very large FE models with more than six million degrees of freedom appear and several hundreds of eigenfrequencies and eigenmodes are needed have shown that AMLS is considerably faster than Lanczos type approaches.

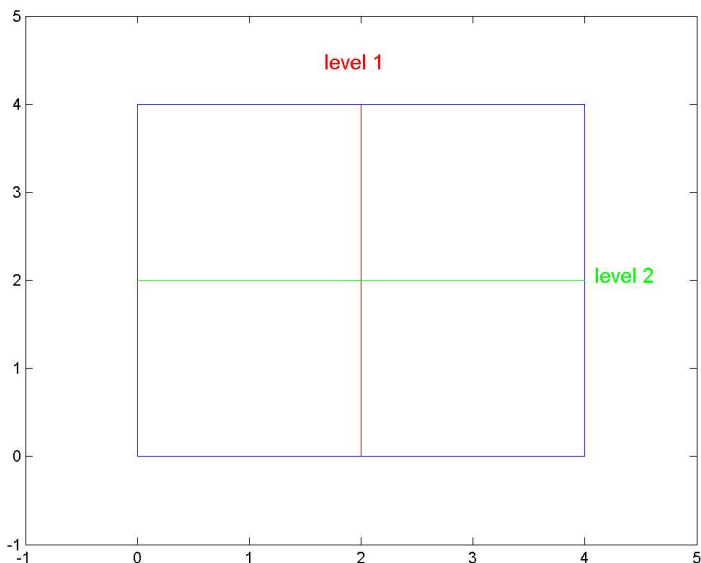
Multi-Level Substructuring: Level 0



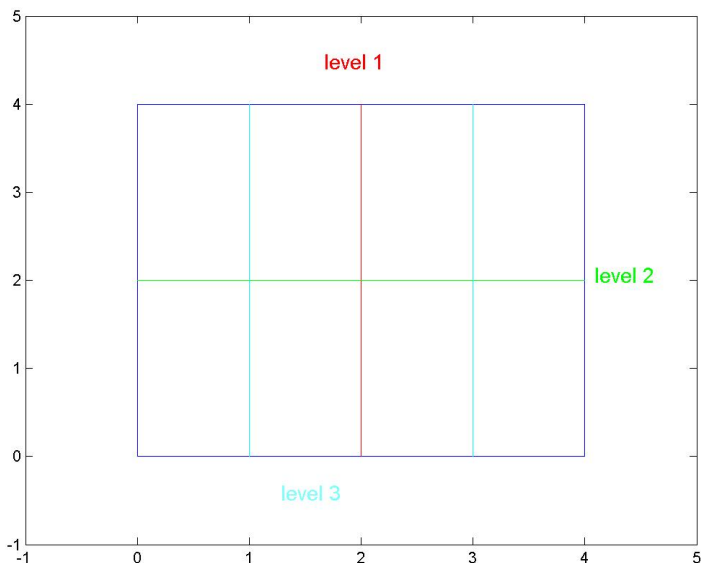
Multi-Level Substructuring: Level 1



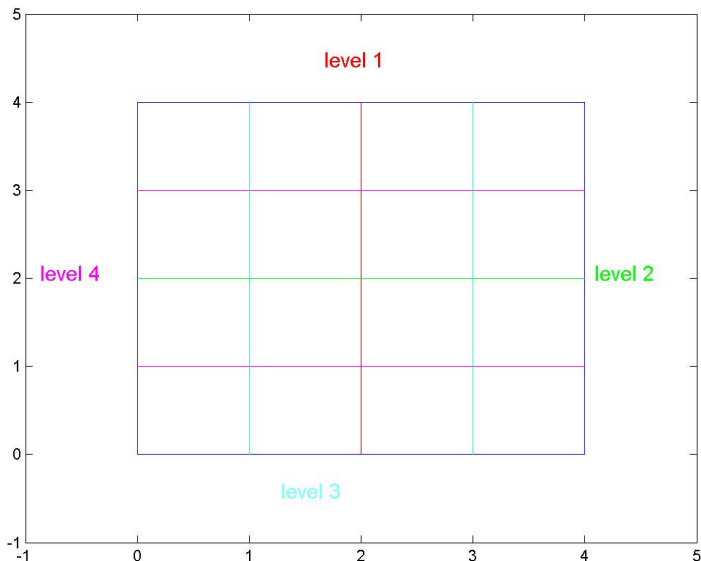
Multi-Level Substructuring: Level 2



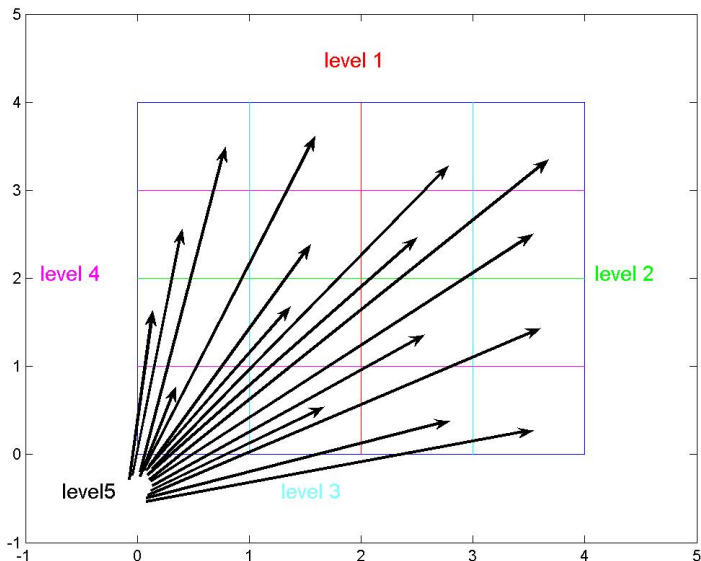
Multi-Level Substructuring: Level 3



Multi-Level Substructuring: Level 4



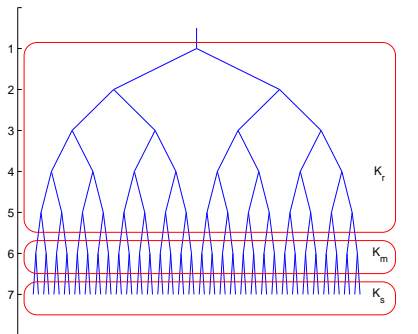
Multi-Level Substructuring: Level 5



AMLS - Algorithm ($Kx = \lambda Mx$)

Reorder System (using Graph Partitioner):

$$\begin{pmatrix} K_s & K_{sm} & K_{sr} \\ K_{sm}^T & K_m & K_{mr} \\ K_{sr}^T & K_{mr}^T & K_r \end{pmatrix} \quad \text{with} \quad K_s = \begin{pmatrix} K_{s_1} & & \\ & \ddots & \\ & & K_{s_n} \end{pmatrix}$$



AMLS - Algorithm ct.

Congruence transformation with

$$U = \begin{pmatrix} I & -K_s^{-1}K_{sr} & -K_s^{-1}K_{mr} \\ O & I & O \\ O & O & I \end{pmatrix}$$

yields

$$\begin{pmatrix} K_s & 0 & 0 \\ 0 & \hat{K}_m & \hat{K}_{mr} \\ 0 & \hat{K}_{mr}^T & \hat{K}_r \end{pmatrix}, \quad \begin{pmatrix} M_s & \hat{M}_{sm} & \hat{M}_{sr} \\ \hat{M}_{sm}^T & \hat{M}_m & \hat{M}_{mr} \\ \hat{M}_{sr}^T & \hat{M}_{mr}^T & \hat{M}_r \end{pmatrix}$$

Notice, that K_s is block-diagonal, and determining $K_s^{-1}K_{sr}$ means that a large number of linear system of small dimension have to be solved. Moreover, the congruence transformation consists of blockmatrix multiplications for blocks of small dimension.

AMLS - Algorithm ct.

Solving of substructure EVPs

$$K_S \Phi_S = M_S \Phi_S \Omega_S, \quad \Phi_S^T M_S \Phi_S = I$$

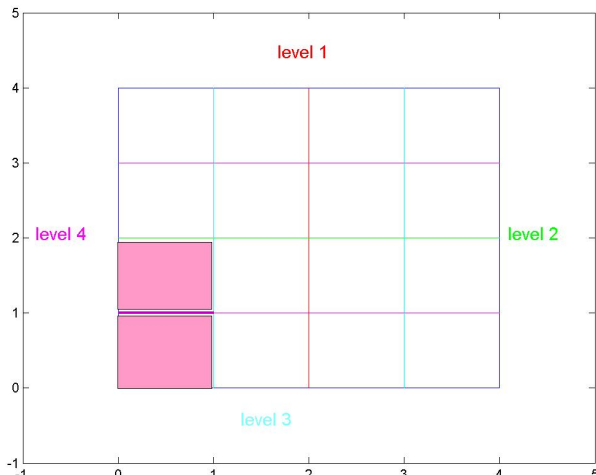
and projecting on a subset of Φ_S (usually corresponding to eigenvalues not exceeding a cut-off frequency) yields

$$\begin{pmatrix} \tilde{\Omega}_S & 0 & 0 \\ 0 & \hat{K}_m & \hat{K}_{mr} \\ 0 & \hat{K}_{mr}^T & \hat{K}_r \end{pmatrix}, \quad \begin{pmatrix} I_s & \tilde{M}_{sm} & \tilde{M}_{sr} \\ \tilde{M}_{sm}^T & \hat{M}_m & \hat{M}_{mr} \\ \tilde{M}_{sr}^T & \hat{M}_{mr}^T & \hat{M}_r \end{pmatrix}$$

This first step of AMLS was introduced already by **Hurty** (1960) and by **Craig** and **Bampton** (1968), and it is called **Component Mode Synthesis (CMS)**.

AMLS - Algorithm ct.

Once substructures on the lowest level have been transformed and reduced by modal projection they are assembled to parent substructures on the next level.



AMLS - Algorithm ct.

Interface and local degrees of freedom are identified, and the substructure models are transformed similarly as on the lowest level.

$$\begin{pmatrix} \tilde{\Omega}_1 & O & O & O \\ O & \tilde{\Omega}_2 & O & O \\ O & O & \tilde{K}_{ij} & \tilde{K}_{ir} \\ O & O & \tilde{K}_{ir}^H & \tilde{K}_{rr} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \lambda \begin{pmatrix} I & O & \tilde{M}_{1i} & \tilde{M}_{1r} \\ O & I & \tilde{M}_{2i} & \tilde{M}_{2r} \\ \tilde{M}_{1i}^H & \tilde{M}_{2i}^H & \tilde{M}_{ij} & \tilde{M}_{ir} \\ \tilde{M}_{1r}^H & \tilde{M}_{2r}^H & \tilde{M}_{ir}^H & \tilde{M}_{rr} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix},$$

Block-elimination of \tilde{K}_{ir} yields

$$\begin{pmatrix} \tilde{\Omega}_1 & O & O & O \\ O & \tilde{\Omega}_2 & O & O \\ O & O & \tilde{K}_{ij} & O \\ O & O & O & \hat{K}_{rr} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \lambda \begin{pmatrix} I & O & \tilde{M}_{1i} & \hat{M}_{1r} \\ O & I & \tilde{M}_{2i} & \hat{M}_{2r} \\ \tilde{M}_{1i}^H & \tilde{M}_{2i}^H & \tilde{M}_{ij} & \hat{M}_{ir} \\ \hat{M}_{1r}^H & \hat{M}_{2r}^H & \hat{M}_{ir}^H & \hat{M}_{rr} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix},$$

AMLS - Algorithm ct.

To perform the modal reduction of the interior degrees of freedom of the current substructure one would have to solve the eigenvalue problem

$$\begin{pmatrix} \tilde{\Omega}_1 & O & O \\ O & \tilde{\Omega}_2 & O \\ O & O & \tilde{K}_{ij} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \omega \begin{pmatrix} I & O & \tilde{M}_{1i} \\ O & I & \tilde{M}_{2i} \\ \tilde{M}_{1i}^H & \tilde{M}_{2i}^H & \tilde{M}_{ij} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}.$$

However, since the number of interior degrees of freedom of substructures grows too large in the course of the algorithm, we reduce the dimension only taking advantage of the eigenvalue problem corresponding to the right lower diagonal block, i.e.

$$\tilde{K}_{ij} \Phi_i = \tilde{M}_{ij} \Phi_i \Omega_i, \quad \Phi_i^H \tilde{M}_{ij} \Phi_i = I.$$

Applying the congruence transformation with $T = \text{diag}\{I, I, \Phi_i, I\}$ and dropping all rows and columns in the third block if the corresponding eigenvalue exceeds the cut-off frequency we further reduce the dimension of the eigenproblem.

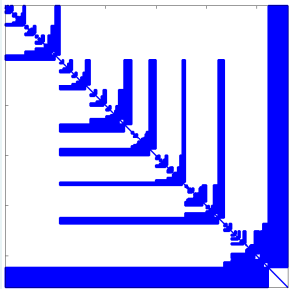
AMLS - Algorithm ct.

Treating coarser levels one after the other in the same way one gets a projected eigenvalue problem of significantly lower dimension

$$K_c x = \lambda M_c x$$

with K_c spd and diagonal and M_c spd in generalized arrowhead structure.

Massmatrix of AMLS



A priori bound (Elsel, V. 2006)

Let $K, M \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, and let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the eigenvalues of problem $Kx = \lambda Mx$, which we assume to be ordered by magnitude.

Let the graph of $|K| + |M|$ be substructured with p levels, and denote by $\tilde{\lambda}_1^{(\nu)} \leq \tilde{\lambda}_2^{(\nu)} \leq \dots$ the eigenvalues obtained by AMLS with cut-off threshold ω_ν on level ν .

If $m \in \mathbb{N}$ such that

$$\lambda_m < \min_{\nu=0, \dots, p} \omega_\nu \leq \lambda_{m+1}$$

then it holds

$$\frac{\tilde{\lambda}_j^{(p)} - \lambda_j}{\lambda_j} \leq \prod_{\nu=0}^p \left(1 + \frac{\tilde{\lambda}_j^{(p)}}{\omega_\nu - \tilde{\lambda}_j^{(p)}} \right) - 1, \quad j = 1, \dots, m.$$

Example

FEM model of 2D problem in vibrational analysis with linear Lagrangean elements.

$n = 68\,862$ degrees of freedom.

Method	10 eigvals	50 eigvals	200 eigvals	
Arnoldi	10.7	37.8	221.1	secs
Jacobi-Davidson	42.2	148.3	901.9	secs

AMLS

ω_c	t_{red}	t_{solve}	n_c	max.err. 10	max.err. 50	max.err. 200
$10 * \lambda_{50}$	205.7	1.7	418	0.14%	3.63%	25.2%
$40 * \lambda_{50}$	209.1	7.5	1407	0.014%	0.37%	2.67%
$50 * \lambda_{50}$	209.2	10.3	1720	0.0097%	0.24%	1.75%
$65 * \lambda_{50}$	211.3	14.7	2166	0.0068%	0.15%	1.05%

Nonlinear eigenproblems

Consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0$$

where $T(\lambda) \in \mathbb{C}^{n \times n}$ is a large and sparse matrix depending on a parameter $\lambda \in D \subset \mathbb{C}$.

As in the linear case $\lambda \in D$ is an eigenvalue of $T(\lambda)x = 0$, if there exists a nontrivial solution $x \neq 0$ which we call a corresponding eigenvector.

Problems of this type arise in vibrations of conservative gyroscopic systems, damped vibrations of structures, problems with retarded argument, lateral buckling problems, fluid-solid vibrations, quantum dot heterostructures, and sandwich plates, e.g.

AMLS for nonlinear eigenproblems

AMLS for linear eigenproblems is a one shot projection method which constructs a suitable subspace by condensation and mode truncation on several levels. This suggests for the nonlinear problem $T(\lambda)x = 0$ the following approach:

Identify (essential) linear part

$$T(\lambda) := -K + \lambda M + R(\lambda),$$

where $K \in \mathbb{C}^{n \times n}$ and $M \in \mathbb{C}^{n \times n}$ are Hermitian and positive definite matrices, and

$$R(\lambda) = K - \lambda M - T(\lambda)$$

is a perturbation of the linear eigenproblem $Kx = \lambda Mx$, which is small in the eigenparameter set of interest.

- apply AMLS to linear pencil (K, M)
- project $T(\lambda)x = 0$ to a search space suggested by first step
- solve projected nonlinear eigenproblem

AMLS for nonlinear eigenproblems ct.

Once the multi-level substructuring transformation of the linear pencil (K, M) has been accomplished with a given cut-off frequency we obtain a matrix Φ_{AMLS} of substructure modes, and a projected eigenproblem

$$\mathcal{K}y = \lambda \mathcal{M}y$$

of much smaller dimension where $\mathcal{K} = \Phi_{\text{AMLS}}^H K \Phi_{\text{AMLS}}$ and $\mathcal{M} = \Phi_{\text{AMLS}}^H M \Phi_{\text{AMLS}}$.

This information can be used in two ways to solve the nonlinear eigenvalue problem approximately:

(1) Project the nonlinear eigenproblem to the subspace of \mathbb{C}^n spanned by substructure modes which were kept in the AMLS reduction, i.e.

$$S(\lambda)y := \Phi_{\text{AMLS}}^H T(\lambda) \Phi_{\text{AMLS}} y = \mathcal{K}y - \lambda \mathcal{M}y - \Phi_{\text{AMLS}}^H R(\lambda) \Phi_{\text{AMLS}} y = 0.$$

In particular this projection can be performed easily if the remainder $R(\lambda)$ has the form

$$R(\lambda) = \sum_{j=1}^p f_j(\lambda) C_j$$

where $f_j(\lambda)$ are given complex functions and $C_j \in \mathbb{C}^{n \times n}$ are given matrices.

AMLS for nonlinear eigenproblems ct.

(2) Determine Ritz pairs $(\lambda_j, x_j) := (\lambda_j, \Phi_{\text{AMLS}} y_j)$, $j = 1, \dots, m$ of the linear problem $Kx = \lambda Mx$ corresponding to eigenvalues in the wanted region, and project the nonlinear problem to the subspace spanned by these Ritz vectors.

Thus we get

$$S(\lambda)z := X^H T(\lambda) Xz = \Lambda z - \lambda z - X^H R(\lambda) Xz = 0$$

where $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_m\}$ and $X = (x_1, \dots, x_m)$.

Projected problem is equivalent to the projection of the AMLS problem of (1) to the space spanned by the eigenvectors y_1, \dots, y_m of $\mathcal{K}y = \lambda \mathcal{M}y$ corresponding to $\lambda_1, \dots, \lambda_m$.

Hence, we can expect, that the first approach will yield better approximations. Examples, however, demonstrate that the loss of accuracy is often negligible.

AMLS for nonlinear eigenproblems ct.

In either case we arrive at a projected nonlinear eigenvalue problem of much smaller dimension which which preserves the structure of the original problem

Depending on the dimension it can be solved by

- a dense solver (based on characteristic function $\det S(\lambda) = 0$, inverse iteration, residual inverse iteration, successive linear problems, e.g.)
- dense method taking advantage of symmetric structure (safeguarded iteration, structure preserving linearization, e.g.)
- iterative projection methods (Arnoldi, Jacobi–Davidson, e.g.)

Gyroscopic eigenproblems

Consider the gyroscopic eigenvalue problem

$$Q(\omega)x := Kx + i\omega Gx - \omega^2 Mx = 0 \quad (1)$$

governing eigenvibrations of rotating structures. Here K is the stiffness matrix modified by the presence of centripetal forces, M is the mass matrix, and G is the gyroscopic matrix stemming from the Coriolis force. Clearly, K and M are symmetric and positive definite, and G is skew-symmetric.

For example, this problem arises when modeling noise of rolling tires which is the major source of traffic noise for passenger cars at speed exceeding 60 km/h. Due to the complicated interior structure of a belted tire the matrices K , M and G of a sufficiently accurate FE model are very large and sparse. Moreover, for the acoustic analysis many eigenpairs not necessarily at the end of the spectrum are needed. Therefore, well-established sparse eigensolvers of Arnoldi type with shift and invert techniques for a linearization or iterative projection methods for nonlinear eigenproblems are very costly since LU factorizations of complex valued matrices $Q(\omega_j)$ for several parameters ω_j are required.

Gyroscopic eigenproblems ct.

Since the influence of the gyroscopic matrix G on the eigenvalues is usually not very high compared to the mass and stiffness matrix, it is reasonable to neglect the linear term $i\omega G$ when defining the essential linear eigenproblem $Kx = \omega^2 Mx$.

Since the sparsity pattern of G matches the ones of K and M one gets the reduced model

$$Ky + i\omega Gy - \omega^2 My = 0,$$

when applying the AMLS reduction to $Kx = \omega^2 Mx$, and projecting the matrix G simultaneously.

Here the stiffness and mass matrix have the same structure as in the linear case, and the gyroscopic matrix \mathcal{G} is a skew-symmetric block matrix containing diagonal blocks corresponding to the (reduced) substructures and interfaces, and only off-diagonal blocks describing the coupling of a substructure and its interface contain non-zero elements. Notice, that all projectors are real, and therefore the reduction can be performed in real arithmetic.

Gyroscopic eigenproblems ct.

If the dimension of the reduced problem is very small, a method at hand is to consider the linearization

$$\begin{pmatrix} i\mathcal{G} & \mathcal{K} \\ \mathcal{K} & \mathbf{O} \end{pmatrix} \begin{pmatrix} \omega y \\ y \end{pmatrix} = \omega \begin{pmatrix} \mathcal{M} & \mathbf{O} \\ \mathbf{O} & \mathcal{K} \end{pmatrix} \begin{pmatrix} \omega y \\ y \end{pmatrix}$$

and to apply any dense solver.

For very large gyroscopic problems (for instance a realistic model of a rolling tire) the dimension of the projected problem will still be quite large. In this case the reduced problem can be solved by an iterative projection method taking advantage of the minmax characterization of its positive eigenvalues or by a sparse solver of the linearized problem like ARPACK. In both cases the solution requires complex arithmetic.

Numerical example

Consider a tire model with 39204 brick elements, 124992 degrees of freedom and 20 different material groups, rotating with 50 km/h. Our aim is to determine approximations to the smallest 180 eigenvalues with relative error less than 1% and the corresponding eigenvectors.

Linearizing in the usual way

$$\begin{pmatrix} -iG & -K \\ I & O \end{pmatrix} \begin{pmatrix} \omega x \\ x \end{pmatrix} = \omega \begin{pmatrix} M & O \\ O & I \end{pmatrix} \begin{pmatrix} \omega x \\ x \end{pmatrix}$$

or by the Hermitian problem

$$\begin{pmatrix} iG & K \\ K & O \end{pmatrix} \begin{pmatrix} \omega x \\ x \end{pmatrix} = \omega \begin{pmatrix} M & O \\ O & K \end{pmatrix} \begin{pmatrix} \omega x \\ x \end{pmatrix}$$

and applying the shift-and-invert Arnoldi method requires an LU factorization of $Q(\omega)$ for every shift ω , which is a complex matrix.

Numerical example ct.

Determining the factorization by SuperLU requires a memory of 6.04 GByte and a CPU time of 3910 seconds on one PA-RISC (750 MHz) processor of an HP superdome.

Applying the nonlinear Arnoldi method the preconditioners can be chosen as real matrices $K - \omega^2 M$, the LU factorization of which requires 2.7 GByte storage and 1940 seconds with SuperLU, and 2.86 GByte storage and 1080 seconds with the multi frontal solver MA57 of HSL.

Since the LU factorization has to be updated several times a total CPU time of more than 12 hours results on one processor of the superdome.

Numerical example ct.

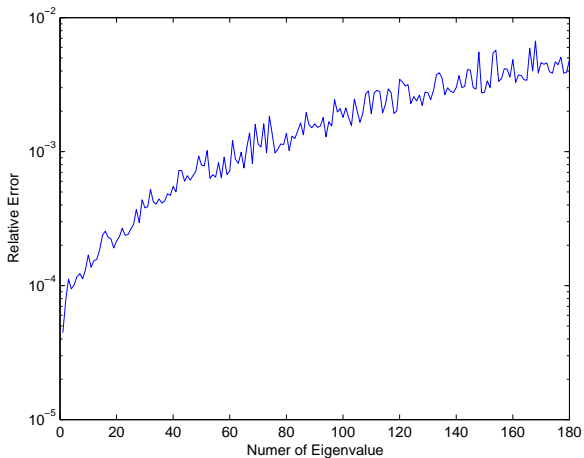
AMLS demands much less storage and the problem under consideration can be solved on a personal computer, namely a Pentium 4 processor with 3.0 GHz and 1 GByte storage.

With a cut-off frequency of $\omega_c = 2 \times 10^5$ the problem is projected to a gyroscopic eigenproblem of dimension $n_c = 2697$ requiring a CPU time of 1187 seconds.

Solving the linearization of the projected problem by `eigs` (i.e. by ARPACK) under MATLAB 7.0 requires another 166.1 seconds.

Numerical example ct.

The relative errors of 180 eigenvalues are all less than 0.67%.



Relativ

Numerical example ct.

Projecting to the subspace spanned by the Ritz vectors corresponding to eigenvalues of the linear problem

$$\mathcal{K}y = \lambda \mathcal{M}y$$

not exceeding $1.5\omega_{\max}^2$ where $\omega_{\max} = 12000$ is the maximal wanted eigenvalue one gets a gyroscopic eigenproblem of dimension 262.

The accuracy of the approximations to the 180 smallest eigenvalues is deteriorated only slightly. The maximum relative error is raised only to 0.69%.

Thus, the solution time of the projected problem is reduced to 39.6 seconds.

Damped vibrations of a structure

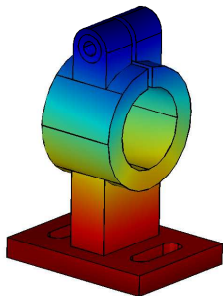
Using a viscoelastic constitutive relation to describe the material behaviour in the equations of motion yields a rational eigenvalue problem in the case of free vibrations.

Discretizing by finite elements yields

$$T(\lambda)x := \left(\omega^2 M + K - \sum_{j=1}^K \frac{1}{1 + b_j \omega} \Delta K_j \right) x = 0$$

where M is the consistent mass matrix, K is the stiffness matrix with the instantaneous elastic material parameters used in Hooke's law, and ΔK_j collects the contributions of damping from elements with relaxation parameter b_j .

Nonproportional damping ct.



FE model of feeder clamp

linear Lagrangean elements on tetrahedral grid

dimension	193'617
nnz(K)	7'670'533
nnz(M)	2'557'851

Determine 30 eigenvalues with maximal negative imaginary part.

Using a viscoelastic constitutive relation to describe the material behaviour in the equations of motion yields a rational eigenvalue problem in the case of free vibrations.

Nonproportional damping ct.

Iterative projection methods

Preconditioner threshold		Arnoldi		Jacobi–Davidson		rational Krylov	
	CPU	# iter.	CPU	# iter.	CPU	# iter.	CPU
10^{-2}	41	601	4137	166	1736	412	9033
10^{-3}	127	226	2004	129	1144	163	3053
10^{-4}	348	109	847	105	1203	78	2268

AMLS

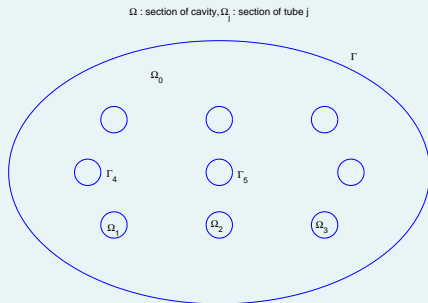
cut-off freq.	CPU time	dim.	CPU solve	max. rel. error
$1.2e7$	837 sec.	1262	7.1 sec.	$2.1e-3$
$2.4e7$	996 sec.	2808	24.1 sec.	$8.1e-4$

Fluid–solid vibrations

Determine acoustic eigenfrequencies of a (very long) cavity containing a tube bundle. Tubes are

- immersed in an inviscid compressible fluid
- rigid, assembled in parallel inside the fluid,
- elastically mounted such that they can vibrate transversally, but can not move in the direction perpendicular to their sections.

Due to these assumptions, three-dimensional effects are neglected, and the problem is studied in any transversal section of the cavity.



Governing problem

Find $\omega > 0$ and $u \in H^1(\omega)$, $u \neq 0$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \omega^2 \int_{\Omega} uv \, dx + \sum_{j=1}^p \frac{\omega^2}{k_j - \omega^2 m_j} \int_{\Gamma_j} un \, ds \cdot \int_{\Gamma_j} vn \, ds$$

for every $v \in H^1(\Omega)$

Discretizing by FEM yields a large and sparse rational eigenvalue problem, e.g. elliptic cavity with 9 tubes of 3 different types

$$T(\lambda)x = (-K + \lambda M + \frac{\lambda}{1-\lambda} C_1 C_1^T + \frac{\lambda}{2-\lambda} C_2 C_2^T + \frac{\lambda}{3-\lambda} C_3 C_3^T)x = 0$$

of dimension $n = 143063$.

Problem has 18, 15, and 14 eigenvalues in the interval $J_1 = (0, 1)$, $J_2 = (1, 2)$, and $J_3 = (2, 3)$, respectively, and a large number of eigenvalues in $(3, \infty)$, 18 of which are contained in $J_4 := (3, 5)$.

Problem ct.

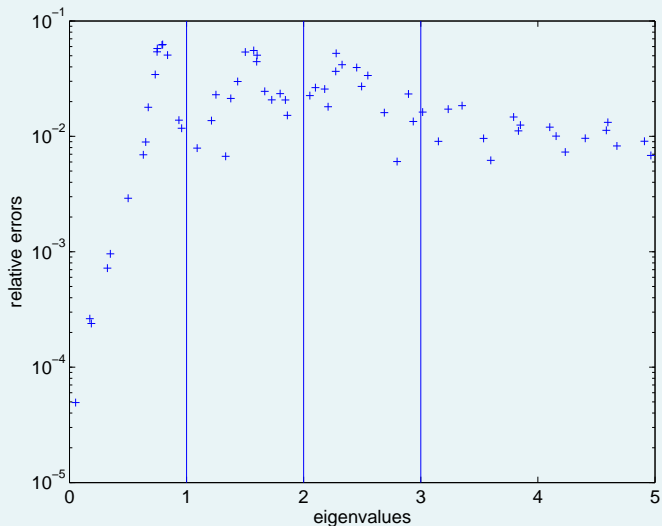
In each of the intervals J_j the eigenvalues can be characterized as minmax values of a Rayleigh functional, and they can be determined one after the other by iterative projection methods.

method	CPU time
Arnoldi	420 sec.
Jacobi-Davidson	1033 sec.

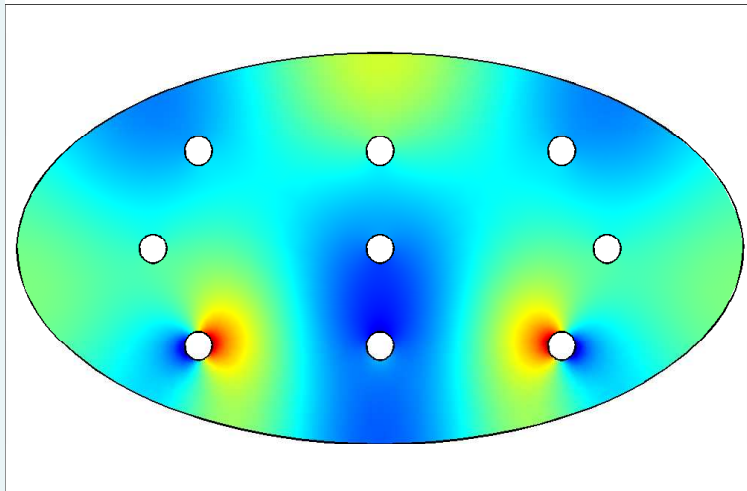
Reducing the problem by AMLS with the base problem $Kx = \lambda Mx$ and a cut-off frequency of 100 generates a rational eigenvalue problem of the same structure of dimension 888 which can be solved by the nonlinear Arnoldi method.

CPU time: Reduction (231 sec.)+Arnoldi (35 sec.) = 266 sec.

Relative errors

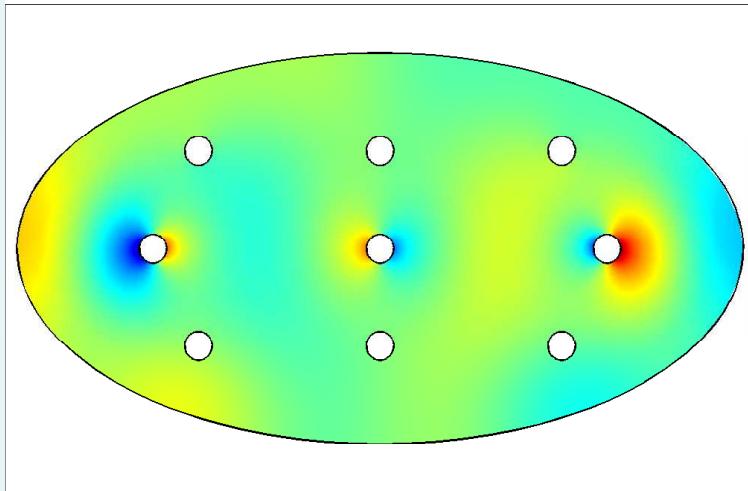


Typical eigenvector

 $\lambda_{13} = 0.7506$ 

Typical eigenvector

$\lambda_{27} = 1.5988$



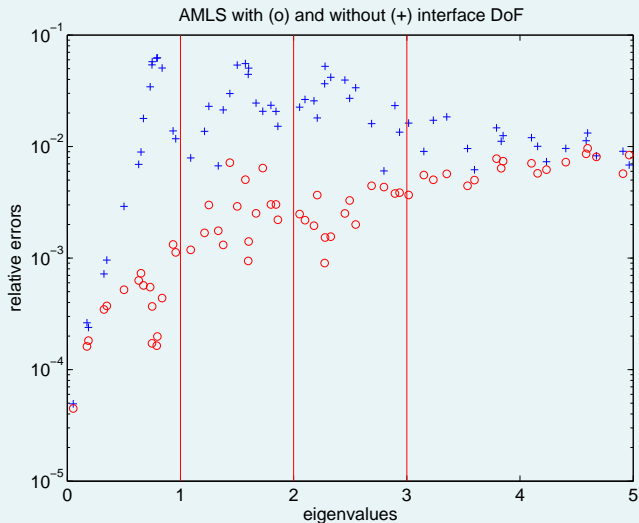
Problem ct.

Complement the 503 interface degrees of freedom on the coarsest level generated by METIS for the base problem $Kx = \lambda Mx$ by the 1728 components corresponding to nonzero row entries of the matrix $[C_1, C_2, C_3]$.

Dimension of the AMLS projected problem: 902

CPU time: Reduction (341 sec.)+Arnoldi (35 sec.) = 376 sec.

Relative errors



Conclusions

- AMLS is a fast method for solving huge linear eigenvalue problems
- AMLS can be generalized to nonlinear eigenproblems identifying an essential linear part of the problem, applying AMLS to this linear problem, and transforming the remaining nonlinear part in the same way
- Special properties of the nonlinear problem have to be taken into account