Stochastic perturbation of scalar conservation laws*

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April 2, 2009

Abstract

In this course devoted to the study of stochastic perturbations of scalar conservation laws, we expose the first part of the paper [EKMS00]. After a short introduction to scalar conservation laws and stochastic differential equations, we give the proof of the existence of an invariant measure for the stoschastic inviscid periodic Burgers' Equation in one dimension according to [EKMS00].

Contents

*Course given at the Spring School *Analytical and Numerical Aspects of Evolution Equations* T.U. Berlin, 30/03/09–03/04/09

1 Scalar conservation laws with source term

Let $\mathbb{T}^N$ be the N-dimensional torus. Let $A \in C^2(\mathbb{R};\mathbb{R}^N)$, $u_0 \in L^\infty(\mathbb{T}^N)$ and $\bar{W} \in C^1(\mathbb{T}^N \times \mathbb{R})$. We consider the equation

$$\partial_t u(x,t) + \operatorname{div}(A(u))(x,t) = \bar{W}(x,t), \quad x \in \mathbb{T}^N, t > 0 \tag{1}$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{T}^N. \tag{2}$$

Eq. (1) is easily solved in the linear case $A(u) = au$, $a \in \mathbb{R}^N$. Indeed, it rewrites as the transport equation $(\partial_t + a \cdot \nabla)u = \bar{W}$, whose solution

$$u(x,t) = u_0(x - ta) + \int_0^t \bar{W}(x - (t-s)a, s)ds$$

satisfies (2).

1.1 Characteristic curves

1.1.1 From solution to characteristic curves

Let $T > 0$. Suppose that $u \in C^1(\mathbb{T}^N \times [0,T])$ is a regular solution to the scalar conservation law (1). By the chain-rule formula, we then have

$$(\partial_t + a(u) \cdot \nabla)u = \bar{W} \text{ in } \mathbb{T}^N \times (0,T), \tag{3}$$

which is a non-linear transport equation with source term. Here $a(u) := A'(u)$. The integral curves γ of the vector field $(1, a(u))^T$ are the curves $(s(\sigma), \xi(\sigma))^T$ satisfying the equation

$$\begin{cases} \dot{s}(\sigma) &= 1, \\ \dot{\xi}(\sigma) &= a(u(\xi(\sigma),\sigma)), \end{cases} \tag{4}$$

and (3) asserts that the derivative of u along any such γ is $\bar{W}(\gamma)$. More technically, for $x \in \mathbb{T}^N$, $t \in (0,T]$, taking $s = \sigma$ (by (4)), let $\xi(s;t,x)$ be the solution to

$$\dot{\xi}(s) = a(u(\xi(s),s)), \quad \xi(t) = x. \tag{5}$$

Set $\nu(s) = u(\xi(s), s)$. By the chain-rule formula, we have

$$\begin{aligned} \dot{\nu}(s) &= \partial_t u(\xi(s), s) + \dot{\xi}(s) \cdot \nabla u(\xi(s), s) \\ &= (\partial_t u + a(u) \cdot \nabla u)(\xi(s), s), \end{aligned}$$

hence, by (3),

$$\dot{\nu}(s) = \bar{W}(\xi(s), s). \tag{6}$$

If furthermore, u satisfies the initial condition (2), then ν satisfies the limit condition

$$\nu(0; t, x) = u_0(\xi(0; t, x)). \tag{7}$$

Moreover, since $\xi(t) = x$, the value at (x, t) of u is $\nu(t; t, x)$. To sum up, we introduce the (time-dependent) vector field

$$\mathbf{V}_s \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} a(v) \\ \bar{W}(x, s) \end{pmatrix}.$$

The equations of characteristics are then the first-order system of ordinary differential equations

$$\begin{pmatrix} \dot{\xi} \\ \dot{\nu} \end{pmatrix} = \mathbf{V}_s \begin{pmatrix} \xi \\ \nu \end{pmatrix} \tag{8}$$

where $\dot{\xi}(s; t, x) = \partial_s \xi(s; t, x)$ and similarly for ν. The ODE has the (non-standard, since coupled) boundary conditions

$$\xi(t; t, x) = x, \quad \nu(0; t, x) = u_0(\xi(0; t, x)), \tag{9}$$

and $\nu(t; t, x) = u(x, t)$.

1.1.2 From characteristic curves to solution

Theorem 1 (From characteristics to solution)**.** *Let $u_0 \in C^1(\mathbb{T}^N)$ and $T > 0$. Assume that there exists $\xi, \nu \in C^1([0, T] \times [0, T] \times \mathbb{T}^N)$ satisfying (8)-(9). Then $u\colon (x, t) \mapsto \nu(t; t, x)$ is a C^1 solution to (1) on $[0, T]$ that satisfies the initial condition (2).*

Remark: The local inversion Theorem applied to the integral form of (8)-(9):

$$\begin{aligned} \xi(s; t, x) &= x - \int_s^t a(\nu(\sigma; t, x)) d\sigma, \\ \nu(s; t, x) &= u_0(\xi(0; t, x)) + \int_0^s \bar{W}(\xi(\sigma; t, x), \sigma) d\sigma, \end{aligned}$$

for $\xi, \nu \in C^0([0, T] \times [0, T] \times \mathbb{T}^N)$ shows that, given $u_0 \in C^1(\mathbb{T}^N)$, there exists a regular solution (ξ, ν) for T small enough (together with Theorem 1, this is precisely the local resolution of (1)-(2) by the method of characteristics).

Proof of Theorem 1: set $u(x,t) = \nu(t;t,x)$. In the coordinates (t,x,v), introduce the surface and section

$$\begin{aligned} \mathscr{S} &= \{(t,x,v) \in [0,T] \times \mathbb{T}^N \times \mathbb{R}; v = u(x,t)\}, \\ \mathscr{S}_t &= \{(x,v) \in \mathbb{T}^N \times \mathbb{R}; v = u(x,t)\}, \quad t \in [0,T]. \end{aligned}$$

We claim that the surface $\mathscr{S}$ can be seen as a "free surface" over of a set of particles that are driven by $\mathbf{V}_t$, *i.e.* $\mathscr{S}_t = \phi_{t,0}(\mathscr{S}_0)$ where $\phi_{t,s}$ is the flow of $\mathbf{V}_t$. Indeed one checks that

$$\begin{pmatrix} \xi(s;t,x) \\ \nu(s;t,x) \end{pmatrix} = \phi_{s,0} \begin{pmatrix} z(x,t) \\ u_0(z(x,t)) \end{pmatrix}, \quad z(x,t) = \pi_x \phi_{0,t} \begin{pmatrix} x \\ u(x,t) \end{pmatrix},$$

where π_x is the projections on the first component x. See Figure 1. The

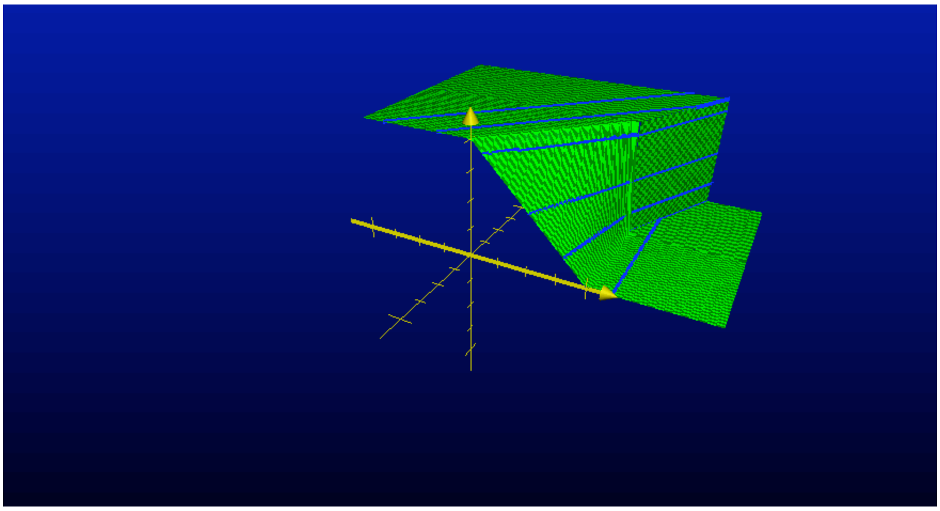

Figure 1: Characteristics and surface $\mathscr{S}$

condition of non-penetration of particles in the free surface then reads

$$\partial_t u - \sqrt{1 + |\nabla u|^2}\mathbf{V}_t \cdot \mathbf{n} = 0 \text{ on } \mathscr{S}, \tag{10}$$

where $\mathbf{n}$ is the out-going unit normal to $\mathscr{S}$,

$$\mathbf{n} = \frac{1}{\sqrt{1 + |\nabla u|^2}} \begin{pmatrix} -\nabla u \\ 1 \end{pmatrix}.$$

Therefore, the condition of non-penetration (10) is

$$\partial_t u + a(v) \cdot \nabla u - \bar{W} = 0 \text{ on } \mathscr{S}.$$

Since $v = u$ on $\mathscr{S}$, we obtain the scalar conservation law (1) in its quasilinear form. ■

Remark: the fact that $\mathscr{S}$ can be seen as a free surface over of a set of particles that are driven by $\mathbf{V}_t$ can also be written as

$$(\partial_t + \mathbf{V}_t \cdot \nabla_{x,v})\mathbf{1}_{\mathscr{D}_t} = 0$$

where $\mathbf{1}_A$ is the characteristic function of a set A and $\mathscr{D}_t = \{(x,v) \in \mathbb{T}^N \times \mathbb{R}; v < u(x,t)\}$ is the subgraph of u. In other words:

$$(\partial_t + a(v) \cdot \nabla_x + \bar{W}(x,t)\partial_v)\mathbf{1}_{u>v} = 0. \tag{11}$$

Eq. (11) can be proven directly on the basis of (3) and the chain rule formula that gives, for $\phi \in C_c^2(\mathbb{R})$,

$$\partial_t \phi(u) + \operatorname{div}\left(\int^u a(v)\phi'(v)dv\right) - \bar{W}\phi'(u) = 0,$$

or

$$\langle(\partial_t + a(v) \cdot \nabla_x + \bar{W}(x,t)\partial_v)\mathbf{1}_{u>v}, \phi'(v)\rangle = 0$$

by use of the identities

$$\int_{\mathbb{R}} \mathbf{1}_{u>v}\phi'(v)dv = \phi(u), \quad \int_{\mathbb{R}} \mathbf{1}_{u>v}a(v)\phi'(v)dv = \int^u a(v)\phi'(v)dv,$$
$$\int_{\mathbb{R}} \mathbf{1}_{u>v}\partial_v\phi'(v)dv = \phi'(u).$$

Eq. (11) is the kinetic formulation of smooth solutions to (1), that will be extended later to non-smooth solutions.

1.1.3 The one-dimensional Burgers' Equation

The (inviscid) Burgers' Equation corresponds to the case $A(u) = \frac{u^2}{2}$ in (1). We also assume that $N = 1$. Note that what follows could be extended more generally to the case of uniformly convex flux A. Indeed, the system of ODEs (8) simplifies as a second order equation in ξ: since $a(v) = v$ is invertible, we obtain $\dot{\xi} = \nu$ and thus $\ddot{\xi} = \bar{W}(\xi,t)$. This last equation is related to the minimization of the functional

$$\mathscr{A}_{0,t}(\xi) := \int_0^t \frac{1}{2}\dot{\xi}(s)^2 + \bar{\mathcal{W}}(\xi(s),s)ds, \quad \bar{\mathcal{W}}(x,t) = \int_0^x \bar{W}(y,t)dy.$$

Indeed, a straightforward computation gives

$$\begin{aligned}
D_\xi \mathscr{A}_{0,t} \cdot \zeta &= \int_0^t \dot{\xi}\dot{\zeta} + \bar{W}(\xi,s)\zeta ds \\
&= \int_0^t \zeta(-\ddot{\xi} + \bar{W}(\xi,s))ds
\end{aligned}$$

for all $\zeta \in C_c^1(0,t)$. Furthermore, the coupled boundary conditions (9) can be easily incorporated in the minimization problem by considering the modified problem (minimization problem with a fixed end)

$$\inf_{\xi(t)=x}\left(\mathscr{A}_{0,t}(\xi)+\int_0^{\xi(0)} u_0(z)dz\right), \tag{12}$$

where the inf is taken over the paths $\xi \in C^1[0,t]$ such that $\xi(t)=x$. Indeed, differentiation of the functional in (12) with respect to $\zeta \in C_c^1[0,t)$ gives, at a point of minimum ξ:

$$\begin{aligned} 0 &= \int_0^t \dot{\xi}\dot{\zeta}+\bar{W}(\xi,s)\zeta ds+u_0(\xi(0))\zeta(0)\\ &= \int_0^t \zeta(-\ddot{\xi}+\bar{W}(\xi,s))ds+[u_0(\xi(0))-\dot{\xi}(0)]\zeta(0), \end{aligned}$$

hence the two equations $\ddot{\xi}=\bar{W}(\xi,t)$ and $\nu(0)=\dot{\xi}(0)=u_0(\xi(0))$ (recall that $\nu=\dot{\xi}$ here). This relation between the Burgers' Equation and the minimization Problem (known as Lax-Oleinik Formula [Lax57, Ole57], and Hopf-Lax Formula in its original context of Hamilton-Jacobi equations) will be fully exploited in the study of the Burgers' Equation with stochastic forcing.

1.2 Entropy solutions

As illustrated on Figure 1 (which corresponds to the Burgers' Equation with initial datum $u_0(x)=\min(1,\max(1-x,0))$ and $x\in\mathbb{R}$ instead of $\mathbb{T}^1$), $\phi_{t,0}(\mathscr{S}_0)$ can cease to be a non-parametrized surface, *i.e.* the graph of a function $u(t)\colon \mathbb{T}^N\to\mathbb{R}$. It is however possible to evolve under $\mathbf{V}_t$ while remaining a graph, as in (11), at the expense of an accurate correction.

Definition 2 (Kinetic solution to scalar conservation laws)**.** *Let $T>0$ and $u_0\in L^\infty(\mathbb{T}^N)$. We say that $u\in L^\infty(\mathbb{T}^N\times(0,T))$ is a (weak) kinetic solution to (1)-(2) if there exists a measurable mapping m from $\mathbb{R}$ into $\mathscr{M}_+(\mathbb{T}^N\times[0,T])$, the space of non-negative measures over $\mathbb{T}^N\times[0,T]$ such that*

$$\begin{cases} m_v := m(v) = 0 \text{ for } |v|>\|u\|_{L^\infty(\mathbb{T}^N\times(0,T))},\\ \int_0^T\int_{\mathbb{T}^N} dm_v(x,t)\leq \|u_0\|_{L^1(\mathbb{T}^N)}, \forall v\in\mathbb{R}, \end{cases}$$

and

$$\int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}\mathbf{1}_{u>v}(\partial_t+a(v)\cdot\nabla_x+\bar{W}(x,t)\partial_v)\varphi dxdtdv+\int_{\mathbb{T}^N}\int_{\mathbb{R}}\mathbf{1}_{u_0>v}\varphi(0)dxdv = \int_0^T\int_{\mathbb{T}^N}\int_{\mathbb{R}}\partial_v\varphi dm_v(x,t)dv \tag{13}$$

for all $\varphi\in C_c^\infty(\mathbb{T}^N\times[0,T)\times\mathbb{R})$.

In other words, up to the initial condition, we have added the right-hand side $\partial_v m$ to (11). Note that (by taking φ independent on v in (13)), we have

$$\partial_t u + \operatorname{div}(A(u)) = \bar{W}$$

in the weak sense.

The kinetic formulation of scalar conservation laws has been developed by Lions, Perthame, Tadmor [LPT94, Per02]. They prove in particular

Theorem 3 (Resolution of Problem (1)-(2))**.** *Let $T > 0$ and $u_0 \in L^\infty(\mathbb{T}^N)$. There exists a unique kinetic solution $u \in L^\infty(\mathbb{T}^N \times (0,T))$ to (1)-(2). Besides, $u \in C([0,T]; L^1(\mathbb{T}^N))$ and, if v is the kinetic solution issued from the initial datum $v_0 \in L^\infty(\mathbb{T}^N)$, we have*

$$\|u(t) - v(t)\|_{L^1(\mathbb{T}^N)} \leq \|u_0 - v_0\|_{L^1(\mathbb{T}^N)}, \quad t \in [0,T].$$

Actually, the solution u given in Theorem 3 is the entropy solution of Problem (1)-(2), according to the following

Definition 4 (Entropy solution to scalar conservation laws)**.** *Let $T > 0$ and $u_0 \in L^\infty(\mathbb{T}^N)$. We say that $u \in L^\infty(\mathbb{T}^N \times (0,T))$ is a (weak) entropy solution to (1)-(2) if for every convex $\eta \in C^2(\mathbb{R})$ and for every non-negative $\theta \in C_c^\infty(\mathbb{T}^N \times [0,T))$,*

$$\int_0^T \int_{\mathbb{T}^N} \eta(u)\partial_t\theta + \Phi(u) \cdot \nabla_x \theta - \bar{W}(x,t)\eta'(u)\theta dxdt + \int_{\mathbb{T}^N} \eta(u_0)\theta(0) dxdv \geq 0 \tag{14}$$

where $\Phi \in C^2(\mathbb{R}; \mathbb{R}^N)$, $\Phi'(v) = a(v)\eta'(v)$.

The two definitions above are equivalent. To go from one formulation to the other, choose $\varphi(x,t,v) = \theta(x,t)\eta'(v)$ in (13) to deduce (14) from (13) and define $\langle m_v, \theta \rangle$ as the left hand-side of (14) with the choice $\eta(u) = (u-v)^+$ to deduce (13) from (14). The notion of entropy solution as defined here goes back to Kruzhkov [Kru70] and is anterior to the notion of kinetic solution.

As a last remark, notice that this theory can be extended to the case where A depends on (x,t): $A = A(x,t,u)$ at least if A has a certain amount of regularity in (x,t) (see section 3.1). We say that a function $u \in L^\infty(\mathbb{T}^N \times (0,T))$ is a (weak) entropy solution to the equation

$$\partial_t u + \operatorname{div}(A(x,t,u)) = \bar{W}$$

with initial datum u_0 if for every convex $\eta \in C^2(\mathbb{R})$ and for every non-negative $\theta \in C_c^\infty(\mathbb{T}^N \times [0,T))$,

$$\int_0^T \int_{\mathbb{T}^N} \eta(u)\partial_t\theta + \Phi(x,t,u)\cdot\nabla_x\theta dxdt \\ - \int_0^T \int_{\mathbb{T}^N} [(\bar{W}(x,t) - (\mathrm{div}A)(x,t,u)\eta'(u) - (\mathrm{div}\Phi)(x,t,u)]\theta dxdt \\ + \int_{\mathbb{T}^N} \eta(u_0)\varphi(0)dxdv \geq 0 \quad (15)$$

where $\partial_v\Phi(x,t,v) = a(x,t,v)\eta'(v)$.

Still anterior to the notion of entropy solution given by Kruzhkov is the notion of entropy solution given by Lax and Oleinik.

1.2.1 Lax-Oleinik Formula

We consider the one-dimensional (inviscid) Burgers' Equation

$$\partial_t + \partial_x\left(\frac{u^2}{2}\right) = \bar{W}(x,t), \quad x \in \mathbb{T}^1, t > 0. \quad (16)$$

Introduce the Skohorod space D of functions $u\colon \mathbb{T}^1 \to \mathbb{R}$ which have discontinuity of the first kind only, *i.e.* for every $x \in \mathbb{T}^1$, $u(x+) := \lim_{y\uparrow x}$ and $u(x-) := \lim_{y\downarrow x} u(y)$ exist and are possibly distinct. We recall that if $u \in D$, then its set of points of discontinuity is countable at most, u is bounded, and $u \in L^\infty(\mathbb{T}^1)$.
Introduce now the action

$$\mathscr{A}_{0,t}(\xi) := \int_0^t \frac{1}{2}\dot{\xi}(s)^2 + \bar{\mathcal{W}}(\xi(s),s)ds, \quad \bar{\mathcal{W}}(x,t) = \int_0^x \bar{W}(y,t)dy.$$

We have the following characterization of entropy solutions.

Theorem 5 (Lax-Oleinik Formula [Lax57, Ole57])**.** *Let $u_0 \in D$ and let $T > 0$. the entropy solution $u \in L^\infty(\mathbb{T}^N \times (0,T))$ to (16) with initial datum u_0 is given by*

$$u(x,t) = \frac{\partial}{\partial x} \inf_{\xi(t)=x} \left\{ \mathscr{A}_{0,t}(\xi) + \int_0^{\xi(0)} u_0(z)dz \right\},$$

where the infimum is taken over all piecewise $C^1[0,t]$ paths with end $\xi(t) = x$, and satisfies $u(t) \in D$ for every $t \in [0,T]$.

2 Stochastic differential equations

We give here a basic summary about stochastic differential equations. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, \beta(t))$ be a stochastic basis, *i.e.*

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space,
- $(\mathcal{F})_{t\geq 0}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$: $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$, $0 \leq s \leq t$,
- $(\beta(t))_{t\geq 0}$ is a brownian motion over $\mathbb{R}$,
- each $\beta(t)$ is $\mathcal{F}_t$-measurable,
- for $0 \leq s \leq t$, $\beta(t) - \beta(s)$ and $\mathcal{F}_s$ are independent.

A real-valued process $(X(t))_{t\geq 0}$ over $(\Omega, \mathcal{F}, \mathbb{P})$ is adapted if each $X(t)$ is $\mathcal{F}_t$-measurable. It is said predictible if it is measurable, as a function of (t, ω), for the σ-algebra generated by the set of products $(s, t] \times F$, $F \in \mathcal{F}_s$, $0 \leq s \leq t$ and by $\{0\} \times \mathcal{F}_0$. Furthermore, a process $(X(t))_{t\geq 0}$ over $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a.s. $C^{0,\alpha}$ $(0 \leq \alpha < 1)$ if for a.e. $\omega \in \Omega$, $t \mapsto X(t, \omega)$ is $C^{0,\alpha}$ (we use the convention $C^{0,0}(\mathbb{R}_+) = C(\mathbb{R}_+)$). It is said to be L^2-continuous if $X\colon \mathbb{R}_+ \to L^2(\Omega)$ is continuous. An L^2-continuous adapted process X has a predictible modification, *i.e.* there exists $\tilde{X}$ a predictible process such that $X(t) = \tilde{X}(t)$ a.e., for every $t \geq 0$. At last, $(X(t))_{t\geq 0}$ is said to be a martingale (*resp.* a sub-martingale) if

$$\mathbb{E}(X(t)|\mathcal{F}_s) = X(s), 0 \leq s \leq t,$$

(*resp.* $\mathbb{E}(X(t)|\mathcal{F}_s) \leq X(s)$) where $\mathbb{E}(X|\mathcal{F}_s)$ denotes the conditional expectation of X with respect to $\mathcal{F}_s$. We have the

Proposition 6 (Doob's inequality)**.** *Let* $(X(t))_{t\geq 0}$ *be an a.s. continuous martingale with* $\mathbb{E}|X(T)|^p < +\infty$*,* $p > 1$*,* $T > 0$*. Then*

$$\mathbb{E}(\sup_{0\leq t\leq T} |X(t)|^p) \leq c_p \mathbb{E}|X(T)|^p, \quad c_p := \left(\frac{p}{p-1}\right)^p. \tag{17}$$

2.0.2 Stochastic integral

For $T > 0$, let $L^2_{\mathcal{P}}(\Omega \times [0,T])$ be the space of predictible $L^2(\Omega \times [0,T])$ processes $(X(t))_{t\in[0,T]}$.

Theorem 7 (Ito's stochastic integral)**.** *There exists an isometry* $L^2_{\mathcal{P}}(\Omega \times [0,T]) \mapsto L^2(\Omega)$ *denoted*

$$Z \mapsto \int_0^T Z(s) d\beta(s)$$

such that

$$\int_0^T Z(s)d\beta(s) = \sum_{k=0}^{n-1} Z_k(\beta(t_{k+1}) - \beta(t_k))$$

if

$$Z = \sum_{k=0}^{n-1} Z_k \mathbf{1}_{(t_k,t_{k+1}]}$$

with $0 = t_0 < t_1 < \cdots < t_n = T$ *and* Z_k *a* $\mathcal{F}_{t_k}$*-measurable random variable. In particular,*

$$\mathbb{E}\left|\int_0^T Z(s)d\beta(s)\right|^2 = \mathbb{E}\int_0^T |Z(s)|^2\, ds, \quad \mathbb{E}\int_0^T Z(s)d\beta(s) = 0. \tag{18}$$

The process $X\colon t \mapsto \int_0^t Z(s)d\beta(s)$ is then an a.s. continous martingale and, by Doob's inequality (17) and Ito's isometry (18),

$$\mathbb{E}(\sup_{0\le t\le T} |X(t)^2|) \le 4\mathbb{E}\int_0^T |Z(s)|^2\, ds.$$

This can be extended to the case where Z is an L^2-continuous adapted process and we denote $dX = Zd\beta$. More generally, if $Y \in L^1(\Omega\times[0,T])$ is adapted, $Z \in L^2_{\mathcal{P}}(\Omega\times[0,T])$, X_0 is $\mathcal{F}_0$-measurable and

$$X(t) = X_0 + \int_0^t Y(s)ds + \int_0^t Z(s)d\beta(s),$$

we denote $dX = Ydt + Zd\beta$ and say that X solves the Problem

$$\begin{cases} dX &= Ydt + Zd\beta, \\ X(0) &= X_0. \end{cases}$$

The Ito Formula can then be stated as follows. Let $u \in C^2_b(\mathbb{R}\times[0,T])$ and $dX = Ydt + Zd\beta$ as above. Then

$$du(X,t) = (u_t(X,t) + u_x(X,t)Y + \frac{1}{2}u_{xx}(X,t)Z^2)dt + u_x(X,t)Zd\beta.$$

All this can be generalized to the cases where β, Y, X take values in $\mathbb{R}^N$ (*resp.* $\mathbb{T}^N$), Z takes values in $\mathbb{R}^{N\times N}$ (*resp.* $\mathbb{Z}^{N\times N}$).

2.0.3 Stochastic differential equation

Let β be a brownian motion over $\mathbb{R}^N$. Let $f \in C_b(\mathbb{R}^N\times[0,T];\mathbb{R}^N)$, $g \in C_b(\mathbb{R}^N\times[0,T];\mathbb{R}^{N\times N})$ be Lipschitz continuous in x, uniformly with respect to t and let X_0 be a $\mathcal{F}_0$-measurable random variable. For the stochastic differential Problem

$$\begin{cases} dX &= f(X,t)dt + g(X,t)d\beta, \\ X(0) &= X_0. \end{cases} \tag{19}$$

we have the

Theorem 8 (The Cauchy Problem for stochastic differential equations)**.** *There exists a unique adapted process $X \in C([0,T]; L^2(\Omega))$ solution to (19). Besides, X has a.s. continuous trajectories.*

2.0.4 Transition semi-group

For $\varphi \in C_b(\mathbb{R}^N)$, $0 \le s \le t$, define

$$P_{t,s}\varphi(x) = \mathbb{E}\varphi(X(t,s,x)), \tag{20}$$

where $X(t,s,x)$ is the value at time t of the solution to $dX = f(X,t)dt + g(X,t)d\beta$ starting at $x \in \mathbb{R}^N$ at time s. Then $P_{t,s}$ is Feller, i.e. $P_{t,s}$ sends $C_b(\mathbb{R}^N)$ in $C_b(\mathbb{R}^N)$ and we have the Chapman-Kolmogorov equation

$$P_{t,\sigma} \circ P_{\sigma,s} = P_{t,s}, \quad s \le \sigma \le t.$$

In the autonomous case $f(X,t) = f(X)$, $g(X,t) = g(X)$, we have $P_{t,s} = P_{t-s,0} =: P_t$ and $P_t P_s = P_{t+s}$, *i.e.* (P_t) is a semi-group over $C_b(\mathbb{R}^N)$. Besides (P_t) is strongly continuous (since f and g are bounded). The generator of (P_t) is

$$u \mapsto f \cdot Du + \frac{1}{2} g^* g : D^2 u.$$

2.0.5 Invariant measure

A probability measure μ on $\mathbb{R}^N$ is said to be invariant if

$$\int_{\mathbb{R}^N} P_{t,s}\varphi(x) d\mu(x) = \int_{\mathbb{R}^N} \varphi(x) d\mu(x)$$

for all $\varphi \in C(\mathbb{R}^N)$ and all $s \le t$. Equivalently, ${P_{t,s}}^*\mu = \mu$ where ${P_{t,s}}^*\mu$ is defined by the duality $\langle {P_{t,s}}^*\mu, \varphi\rangle = \langle \mu, P_{t,s}\varphi\rangle$, $\varphi \in C_b(\mathbb{R}^N)$. Note that, if X is solution to (19), and μ_t is the law of $X(t)$, then, for $\varphi \in C_b(\mathbb{R}^N)$, and $s \le t$,

$$\begin{aligned} \int_{\mathbb{R}^N} \varphi(x) d\mu_t(x) &= \mathbb{E}\varphi(X(t)) \\ &= \mathbb{E}\varphi(X(t,s,X(s))) \\ &= (P_{t,s}\varphi)(X(s)) \\ &= \int_{\mathbb{R}^N} (P_{t,s}\varphi)(x) d\mu_s(x). \end{aligned}$$

Consequently, we have the following

Proposition 9 (Invariant solution and invariant measure). *Assume X is an invariant solution, i.e. the law μ of X does not depend on t. Then μ is an invariant measure.*

Hence, existence of invariant solution gives the existence of an invariant measure. An other way to exhibit invariant measures is to use the Krylov-Bogoliubov Theorem and the Prohorov Theorem.

Theorem 10 (Prohorov Theorem). *Let E be a metric space. If a sequence (μ_n) of probability measure is tight, i.e. for all $\varepsilon > 0$, there exists a compact K_ε in E such that $\mu_n(K_\varepsilon) > 1-\varepsilon$ for all n then there is a subsequence (μ_{n_k}) which is weakly converging to a probability measure μ: for all $\varphi \in C_b(E)$,*

$$\lim_{k\to+\infty} \int_E \varphi(x) d\mu_{n_k}(x) = \int_E \varphi(x) d\mu(x).$$

We consider the autonomous case in (19).

Theorem 11 (Krylov-Bogoliubov). *Let μ be a probability measure on $\mathbb{R}^N$. If, for some probability measure ν and for some sequence $(T_n) \uparrow +\infty$, the sequence $\frac{1}{T_n}\int_0^{T_n} P_t^* \nu dt$ converges weakly to μ, then μ is an invariant measure.*

Corollary 12. *Suppose that, for some probability measure ν, the family $\left\{\frac{1}{T}\int_0^T P_t^*\nu dt; T > 0\right\}$ is tight. Then there exists en invariant measure.*

2.0.6 Examples

1. ODE: For the autonomous equation $\dot{X} = f(X)$ (with, say, f continuously differentiable and bounded), the invariant measures are the (convex combinations of the) Dirac masses δ_x, $x \in f^{-1}(0)$.

2. Langevin Equation: We consider the stochastic differential equation

$$dX = -bX dt + \sigma d\beta, \quad b > 0, \sigma \in \mathbb{R} \setminus \{0\}. \tag{21}$$

By Ito Formula,

$$\frac{1}{2} d|X|^2 = -b|X|^2 dt + \sigma X d\beta + \frac{\sigma^2}{2} dt,$$

and

$$\frac{1}{2}\mathbb{E}|X|^2(T) + b\int_0^T \mathbb{E}|X(t)|^2 dt = \frac{1}{2}\mathbb{E}|X(0)|^2 + \frac{\sigma^2}{2} T.$$

In particular, taking $\nu =$ law of $X(0)$, so that $P_t^*\nu =$ law of $X(t)$, we have, for $\mu_T := \frac{1}{T}\int_0^T P_t^*\nu$,

$$\int_{\mathbb{R}^N} |x|^2 d\mu_T(x) \leq \frac{C}{b}.$$

Considering the compact $K_R = \overline{B}(0, R)$, we deduce

$$\mu_T(K_R^c) \leq \frac{1}{R^2} \int_{|x|>R} |x|^2 d\mu_T(x) \leq \frac{C}{bR^2},$$

hence $\mu_T(K_R) \geq 1 - \frac{C}{bR^2}$ for all T, which shows that $\{\mu_T\}_{T>0}$ is tight. By corollary 12, there exists an invariant measure for (21) (actually, this invariant measure is unique and is a Gaussian).

3. Stochastic Heat Equation: We consider the stochastic partial differential equation

$$dX(t) = b\Delta X(t)dt + dW \text{ in } \mathbb{T}^N, \quad b > 0. \tag{22}$$

Let $e_n \colon x \mapsto e^{2\pi i n \cdot x}$, $n \in \mathbb{Z}^N$, define the canonical Fourier basis on $\mathbb{T}^N$. Let $(\beta_n(t))_{n\in\mathbb{Z}^d}$ be some independent brownian motion over $\mathbb{R}$. We assume that W is the cylindrical Wiener process defined by

$$W(t) = \sum_{n\in\mathbb{Z}^N} \sigma_n \beta_n(t) e_n, \quad (\sigma_n) \in l^2(\mathbb{Z}^N), \tag{23}$$

so that (22) can be seen as the countable collection of Langevin Equations for $X_n := \langle X, e_n\rangle$, $X \in L^2(\mathbb{T}^N)$:

$$dX_n(t) = -4\pi^2 b|n|^2 X_n dt + \sigma_n d\beta_n(t). \tag{24}$$

All the following computations (and the fact that P_t is Feller) can be justified by using (24). By Ito Formula, we have

$$\begin{aligned}\frac{1}{2}\mathbb{E}\|X(T)\|^2_{L^2(\mathbb{T}^N)} + b\int_0^T \mathbb{E}\|\nabla X(t)\|^2_{L^2(\mathbb{T}^N)} dt \\ = \frac{1}{2}\mathbb{E}\|X(0)\|^2_{L^2(\mathbb{T}^N)} + \frac{\|\sigma\|^2_{l^2(\mathbb{Z}^N)}}{2} T.\end{aligned} \tag{25}$$

In particular, we have the bound

$$\frac{1}{T}\int_0^T \mathbb{E}\|X(t)\|^2_{H^1(\mathbb{T}^N)} dt \leq \frac{C}{b}. \tag{26}$$

Take ν = law of $X(0)$, so that $P_t^*\nu$ = law of $X(t)$ and set $\mu_T = \frac{1}{T}\int_0^T P_t^*\nu$. We have

$$\int_{L^2(\mathbb{T}^N)} \|u\|^2_{H^1(\mathbb{T}^N)} d\mu_T(u) \leq \frac{C}{b},$$

hence, for $K_R := \{u \in L^2(\mathbb{T}^N); \|u\|_{H^1(\mathbb{T}^N)} \leq R\}$,

$$\mu_T(K_R^c) \leq \frac{1}{R^2} \int_{L^2(\mathbb{T}^N)} \|u\|^2_{H^1(\mathbb{T}^N)} d\mu_T(u) \leq \frac{C}{bR^2}$$

and $\mu_T(K_R) \geq 1 - \frac{C}{bR^2}$, which proves that $\{\mu_T\}_{T>0}$ is tight, and gives the existence of an invariant measure by Corollary 12.

4. Stochastic parabolic Equation: We consider the stochastic partial differential equation

$$dX(t) = \operatorname{div}(A(X))dt + b\Delta X(t)dt + dW \text{ in } \mathbb{T}^N, \quad b > 0 \tag{27}$$

with W as in (23) and $A \in C^2(\mathbb{R};\mathbb{R}^N)$. Since $\langle \operatorname{div}(A(X)), X\rangle_{L^2} = 0$, Ito Formula again gives (25) as above, hence the existence of an invariant measure.

In all the examples above (except the one of deterministic ODEs), the existence of an invariant measure is ensured by a dissipation mechanism contained in the ODE or PDE, and indicated in the fact that $b > 0$ (observe that all the bounds that give tightness, as (26) for example, depend on b^{-1}). What happen when $b = 0$?

3 Scalar conservation laws with stochastic forcing

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis. We consider the first-order scalar conservation law with stochastic forcing

$$du + \operatorname{div}(A(u))dt = dW, \quad x \in \mathbb{T}^N, t > 0. \tag{28}$$

We suppose that the noise has the structure

$$W = \operatorname{div}(\mathcal{W}), \quad \mathcal{W}(x,t) = \sum_{i=k}^{\infty} F_k(x)\beta_k(t)$$

where the $\beta_k(t)$s are independent brownian motion on $\mathbb{R}$ the F_k a regular functions, $F_k \in C^3(\mathbb{T}^N;\mathbb{R}^N)$ with

$$\sum_{k\geq 1} \|F_k\|_{C^3(\mathbb{T}^N;\mathbb{R}^N)} < +\infty.$$

The flux function A in (28) is supposed to satisfy $A \in C^2(\mathbb{R};\mathbb{R}^N)$. Eq. (28) will be appended with a (deterministic) initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{T}^N, \quad u_0 \in L^\infty(\mathbb{T}^N). \tag{29}$$

3.1 Entropy solution

To define a notion of entropy solution to (28), we take advantage of the fact that the noise is additive (independent on u) to rewrite the equation as

$$d(u - W) + \operatorname{div}(A(u)) = 0,$$

i.e.

$$\partial_t u_W + \mathrm{div}(B(x,t,u_W)) = 0, \quad B(x,t,u_W) := A(u_W + W(x,t)), \qquad (30)$$

for the new unknown $u_W := u - W$. For a.e. ω, $W^\omega \in C_t^{0,1/3} C_x^3$, hence $B(\cdot, u) \in C_t^{0,1/3} C_x^3$. This is enough regularity to solve the Problem (30)-(29). By use of an adequate notion of generalized solution, one can prove the following result.

Definition 13 (Entropy solution)**.** *Let $T > 0$ and $u_0 \in L^\infty(\Omega)$. We say that $u \in L^2(\mathbb{T}^N \times (0,T) \times \Omega)$ is solution to (28)-(29) if $u(t)$ is adapted and if, for a.e. ω, $u^\omega - W^\omega$ is an entropy solution to (30)-(29) in the sense of (15).*

Theorem 14. *[VW09] For any $u_0 \in L^\infty(\Omega)$, for any $T > 0$, there exists a unique entropy solution $u \in L^2(\mathbb{T}^N \times (0,T) \times \Omega)$ to (28)-(29).*

We also refer to [Kim03] and to [FN08] for the case of a noise depending on the unknown u (multiplicative noise, $1D$).

3.2 Burgers' Equation

Assume $N = 1$, $A(u) = u^2/2$. Then (28) is the (inviscid) Burgers' Equation with stochastic forcing

$$du + \partial_x(u^2/2)dt = dW(t). \qquad (31)$$

For (31), the Lax-Oleinik formula given in Section 1.2.1 holds. Actually, the term related to the source term in the action $\mathscr{A}_{\tau,t}$ is now

$$\int_\tau^t \sum_{k\geq 1} F_k(\xi(s))d\beta_k(s).$$

To avoid the use of the stochastic integral, we use integration by parts to rewrite it as

$$-\int_\tau^t \sum_{k\geq 1} f_k(\xi(s))\dot\xi(s)(\beta_k(s)-\beta_k(\tau))ds + \sum_{k\geq 1} F_k(\xi(t))(\beta_k(t)-\beta_k(\tau)),\, f_k := F_k'.$$

If $\xi(t)$ is fixed to be x, then the term $F_k(\xi(t))(\beta_k(t) - \beta_k(\tau))$ is independent on ξ, hence we redefine the action as

$$\mathscr{A}_{\tau,t}(\xi) = \int_\tau^t \frac{1}{2}|\dot\xi(s)|^2 - \sum_{k\geq 1} f_k(\xi(s))\dot\xi(s)(\beta_k(s) - \beta_k(\tau))ds, \qquad (32)$$

for $\xi \in C^1[\tau, t]$.

Remark: in all the analysis that follows we will consider minimizers of the action $\mathscr{A}_{s,t}$ for fixed ω, hence this latter should be denoted $\mathscr{A}^\omega_{s,t}$, which we will not do for brevity of notations.

Theorem 15. *[EKMS00] Let $u_0 \in D$ and let $t_0 \in \mathbb{R}$ and $T > t_0$. Let the action $\mathscr{A}_{\tau,t}$ be defined by (32). The entropy solution $u \in L^\infty(\mathbb{T}^N \times (t_0, T))$ to (31) with initial datum u_0 at $t = t_0$ is given by*

$$u^\omega(x,t) = \frac{\partial}{\partial x} \inf_{\xi(t)=x} \left\{ \mathscr{A}_{t_0,t}(\xi) + \int_0^{\xi(t_0)} u_0(z)dz \right\}, t > t_0$$

for a.e. ω. Besides, for a.e. ω, for every $t \in [t_0, T]$, $u^\omega(t) \in D$.

Notice that we have considered the solution to (31) starting at $t_0 \in \mathbb{R}$, not necessarily $t_0 = 0$. It is straightforward to generalized the definition of entropy solution given in Def. 13 to this setting. However it supposes that the brownian processes $\beta_k(t)$ are defined for $-\infty < t < +\infty$. This can be done by inverting the Markov process $(\beta_k(t))_{t\geq 0}$ with respect to an invariant measure and precisely, here, with respect to the Lebesgue measure on $\mathbb{T}^1$.

4 Invariant measure for the stochastic Burgers' Equation

We will now follow § 1,2,3,4 in [EKMS00] to prove the following

Theorem 16. *The stochastic inviscid Burgers' Equation (31) admits an invariant measure.*

We have seen above (Example 4. in Section 2.0.6) that stochastic parabolic equations admit an invariant measure because of dissipation mechanisms. For the inviscid Burgers' Equation, the dissipation (of the energy supplied by the source term) occur in shocks. Actually, shocks result from the non-linear character of the flux $u \mapsto u^2/2$.

4.1 One-sided minimizers

To construct an invariant measure, we will construct an invariant solution (*cf* Prop. 9). To this aim we will show that, for a.e. ω, there exists a solution u^ω starting from $u_0 \equiv 0$ at $t_0 = -\infty$. This solution will be build *via* minimizers of the action $\lim_{t_0 \to -\infty} \mathscr{A}_{t_0,0}$. More properly, we will consider one-sided minimizers according to the following definition.

Definition 17 (One-sided minimizer)**.** *Let $t \in \mathbb{R}$. A piecewise C^1-curve $\xi \colon (-\infty, t] \to \mathbb{T}^1$ is a one-sided minimizer if*

$$\mathscr{A}_{s,t}(\xi) \leq \mathscr{A}_{s,t}(\tilde{\xi})$$

for any compact perturbation $\tilde{\xi} \in \mathrm{Lip}(-\infty, t]$ such that $\tilde{\xi}(t) = \xi(t)$, $\tilde{\xi} = \xi$ on $(-\infty, \tau]$, and any $s \leq \tau$.

Remark (regularity of minimizers): for a.e. ω, $\beta_k(\cdot, \omega) \in C^{0,1/3}(\mathbb{R})$ for all k. For such ω, any minimizer (among piecewise C^1-curves) of the action $\mathscr{A}_{t_1,t_2}$, $t_1 < t_2$ will be actually in $C^{1,1/3}[t_1, t_2]$. This can be seen by observing that

$$\frac{1}{4}\int_{t_1}^{t_2} |\dot{\xi}(s)|^2 ds - \sum_{k \geq 1} \|f_k\|_{C(\mathbb{T}^1)} \int_{t_1}^{t_2} |\beta_k(s) - \beta_k(t_1)| ds \leq \mathscr{A}_{t_1,t_2}(\xi), \tag{33}$$

which shows that any minimizer is in $H^1(t_1, t_2)$, then by writing the Euler equation for the minimization of $\mathscr{A}_{t_1,t_2}$, that gives

$$\begin{aligned} \dot{\xi}(\tau) = \dot{\xi}(s) + \sum_{k \geq 1} f_k(\xi(\tau))(\beta_k(\tau) - \beta_k(t_2)) - f_k(\xi(s))(\beta_k(s) - \beta_k(t_2)) \\ - \sum_{k \geq 1} \int_s^\tau f_k'(\xi(\sigma))\dot{\xi}(\sigma)(\beta_k(\sigma) - \beta_k(t_2)) d\sigma, \end{aligned} \tag{34}$$

for a.e. $t_1 < s < \tau < t_2$.

In relation with the regularity of minimizers, we have the following

Proposition 18. *Let $s < \tau < t$ and let $\xi \in C([s,t]) \cap (C^1([s,\tau]) \cup C^1([\tau,t]))$ have a corner at τ: $\dot{\xi}(\tau-) \neq \dot{\xi}(\tau+)$. Then, by smoothing of ξ, the action $\mathscr{A}_{s,t}(\xi)$ is strictly decreased.*

Proof: Let ξ^ε be a regularization of ξ around $\sigma = \tau$. We can suppose that ξ^ε and ξ have same ends, in which case $\mathscr{A}_{s,t}(\xi^\varepsilon) - \mathscr{A}_{s,t}(\xi) = \mathscr{A}'_{s,t}(\xi^\varepsilon) - \mathscr{A}'_{s,t}(\xi)$, where

$$\mathscr{A}'_{s,t}(\xi) = \int_s^t \frac{1}{2}|\dot{\xi}(\sigma)|^2 - \sum_{k \geq 1} f_k(\xi(\sigma))\dot{\xi}(\sigma)(\beta_k(\sigma) - \beta_k(\tau)) d\sigma.$$

Since $|\beta_k(\sigma) - \beta_k(\tau)| = \mathcal{O}(\varepsilon^{1/3})$ for $|\sigma - \tau| < \varepsilon$, this is the first term in $\mathscr{A}'_{s,t}$ that will be dominant in the comparison of $\mathscr{A}'_{s,t}(\xi^\varepsilon)$ with $\mathscr{A}'_{s,t}(\xi)$. It is obvious that, by right or left smoothing of ξ, according to the sign of $\dot{\xi}(\tau+) - \dot{\xi}(\tau-)$, one strictly decreases $\|\dot{\xi}\|_{L^2}$. ■

Remark (restriction of minimizers): Observe that, for $s \leq \tau \leq t$, we have $\mathscr{A}_{s,\tau} + \mathscr{A}_{\tau,t} = \mathscr{A}_{s,t} + R_{s,\tau,t}$ with

$$R_{s,\tau,t}(\xi) := \sum_{k \geq 1} [F_k(\xi(t)) - F_k(\xi(\tau))][\beta_k(\tau) - \beta_k(s)].$$

In particular, if γ is a minimizer of $\mathscr{A}_{s,t}$, then, for $[s', t'] \subset [s, t]$, γ is a minimizer of $\mathscr{A}_{s',t'}$ *among* all piecewise C^1 curves having *same* ends as γ at s' and t'.

4.1.1 Bound on velocities

Lemma 19. *For a.e. ω and any $t \in \mathbb{R}$, there exists $T(\omega, t)$ and $C(\omega, t)$ such that, if γ minimizes $\mathscr{A}_{t_1,t}$ and $t_1 < t - T(\omega, t)$, then*

$$|\dot\gamma(t)| \le C(\omega, t).$$

Remark: in the case $F_k \equiv 0$ (no noise), the minimizers of the action are straight lines :

$$\dot\xi(s) = \frac{\xi(t) - \xi(t_1)}{t - t_1}.$$

In particular, since $|\xi(t) - \xi(t_1)| \le 1$, $|\dot\xi(t)| \le 1$ as soon as $t_1 < t - 1$. The proof of Lemma 19 rests on this simple fact together with a perturbation argument.

Proof of Lemma 19: Set

$$C_1(\omega, t) = \frac{1}{4} + \max_{t-1 \le s \le t} \sum_{k \ge 1} \|F_k\|_{C^2(\mathbb{T}^1)} |\beta_k(s) - \beta_k(t)|$$

and $T(\omega, t) = \frac{1}{4C_1(\omega,t)}$, $C(\omega, t) = 20C_1(\omega, t)$. Set also

$$v_* = |\dot\gamma(t)|, \quad v_M = \max_{t-T \le s \le t} |\dot\gamma(s)|, \quad v_m = \min_{t-T \le s \le t} |\dot\gamma(s)|.$$

We first show a Harnack inequality:

$$v_* > 16C_1 \Rightarrow \frac{1}{2} v_* \le v_m \le v_M \le \frac{3}{2} v_*. \tag{35}$$

Indeed, by modification of (34), we have, for $s \in [t - T, t]$,

$$\dot\gamma(s) = \dot\gamma(t) + \sum_{k \ge 1} f_k(\gamma(s))(\beta_k(s) - \beta_k(t)) + \int_s^t \dot\gamma(\sigma) \sum_{k \ge 1} f_k'(\gamma(\sigma))(\beta_k(\sigma) - \beta_k(t)) d\sigma,$$

hence

$$\begin{aligned} |\dot\gamma(s)| &\le v_* + C_1 + C_1 T v_M \\ &\le v_* + C_1 + \frac{1}{4} v_M. \end{aligned}$$

It follows $v_M \le v_* + C_1 + \frac{1}{4} v_M$ and $v_M \le \frac{4}{3}(v_* + C_1)$, whence the inequality $v_M \le \frac{3}{2} v_*$ in (35). Similarly, we have

$$|\dot\gamma(s) - \dot\gamma(t)| \le C_1 + \frac{1}{4} v_M$$

and, if $v_* > 16C_1$,

$$|\dot\gamma(s) - \dot\gamma(t)| \le \frac{1}{16}v_* + \frac{3}{8}v_* \le \frac{1}{2}v_*,$$

which gives the second inequality in (35). Now, if ξ is a C^1-curve having the same extremities as γ at $t-T$ and t, we have

$$\begin{aligned} 0 &\le \mathscr{A}_{t-T,t}(\xi) - \mathscr{A}_{t-T,t}(\gamma) \\ &= \frac{1}{2}\int_{t-T}^{t}(|\dot\xi(s)|^2 - |\dot\gamma(s)|^2)ds \\ &\quad -\sum_{k\ge 1}[f_k(\xi(s))\dot\xi(s) - f_k(\gamma(s))\dot\gamma(s)](\beta_k(s)-\beta_k(t))ds \\ &\le \frac{T}{2}(\max_{t-T\le s\le t}|\dot\xi(s)|^2 - v_m^2) + C_1T(\max_{t-T\le s\le t}|\dot\xi(s)| + v_M). \end{aligned}$$

Taking for ξ the straight line joining $\gamma(t-T)$ to $\gamma(t)$, we have $|\dot\xi(s)| \le T^{-1}$, hence

$$\begin{aligned} 0 \le \mathscr{A}_{t-T,t}(\xi) - \mathscr{A}_{t-T,t}(\gamma) &\le \frac{1}{2}(\frac{1}{T} - Tv_m^2) + \frac{1}{4}(\frac{1}{T} + v_M) \\ &\le \frac{3}{4T} - \frac{T}{2}v_m^2 + \frac{1}{4}v_M. \end{aligned} \tag{36}$$

If $v_* > 16C_1$, then (35) and (36) give $0 \le \frac{3}{4T} - \frac{T}{8}v_*^2 + \frac{1}{4}v_*$, which implies $v_* \le 20C_1$. This concludes the proof of the lemma. ■

Corollary 20. *For a.e. ω and any $t_1 < t' < t$, there exists $T(\omega,t)$ and $C(\omega,t,t')$ such that, if γ minimizes $\mathscr{A}_{t_1,t}$ and $t_1 < t - T(\omega,t)$, then, for every $s \in [t',t]$,*

$$|\dot\gamma(s)| \le C(\omega,t,t').$$

Proof: By Lemma 19, we have $|\dot\gamma(t)| \le C(\omega,t)$. Let $\tilde\gamma$ be the straight line joining $\gamma(t')$ to $\gamma(t)$. Using $\mathscr{A}_{t',t}(\gamma) \le \mathscr{A}_{t',t}(\tilde\gamma)$ in (33) gives $\|\dot\xi\|_{L^2(t',t)} \le C(\omega,t,t')$. Reporting in (34) where $s=t$, we obtain $|\dot\gamma(\tau)| \le C(\omega,t,t')$ (for possibly a different constant), $\tau \in [t',t]$. ■

Lemma 21 (Limit of minimizers)**.** *For $x \in \mathbb{T}^1, t \in \mathbb{R}$, let (γ_n) be a sequence of minimizers of the action $\mathscr{A}_{t_n,t}$ with fixed end $\gamma_n(t) = x$. Suppose that $t_n \to -\infty$ and that $\gamma_n \to \gamma \in C^1(-\infty,t]$ for the C^1-convergence on every compact subset of $(-\infty,t]$. Then γ is a one-sided minimizer with end $\gamma(t) = x$.*

Proof: Let $\xi \in C^1(-\infty,t]$ satisfy $\xi(t) = x$, $\xi = \gamma$ on $(-\infty,\tau]$. Fix $s \le \tau$ and let (ξ_n) be a sequence of $C^1[s,t]$-curves such that ξ_n and γ_n have same extremities and $\xi_n \to \xi$ in $C^1[s,t]$. Such a (ξ_n) exists since $\gamma_n(s) \to \gamma(s) = \xi(s)$. We have $\mathscr{A}_{s,t}(\gamma) = \lim_{n\to+\infty} \mathscr{A}_{s,t}(\gamma_n)$ and, for $t_n \le s$, $\mathscr{A}_{s,t}(\gamma_n) \le \mathscr{A}_{s,t}(\xi_n)$. Since $\mathscr{A}_{s,t}(\xi_n) \to \mathscr{A}_{s,t}(\xi)$, it follows that $\mathscr{A}_{s,t}(\gamma) \le \mathscr{A}_{s,t}(\xi)$. ■

As a corollary of Corollary 20 and Lemma 21, we obtain the existence of one-sided minimizers.

Theorem 22 (Existence of one-sided minimizers)**.** *For a.e.* ω*, for every* $x \in \mathbb{T}^1$*,* $t \in \mathbb{R}$*, there exists a one-sided minimizer* γ *such that* $\gamma(t) = x$*.*

Proof: For fixed ω, x, t, let γ_n be a minimizer of $\mathscr{A}_{-n,t}$ satisfying $\gamma_n(t) = x$ (that γ_n exists results from (33)). For $n > T(t,\omega) - t$, and $-n < s < t$, we have $\|\dot\gamma_n\|_{C[s,t]} \le C(\omega, t, s)$ by Corollary 20, hence, up to a subsequence, there exists $\gamma \in H^1(s,t)$ such that $\gamma_n \to \gamma$ in $C[s,t]$ and $\dot\gamma_n \rightharpoonup \dot\gamma$ in $L^2(s,t)$. From (34), it follows that, possibly after a new extraction, $\gamma_n \to \gamma$ in $C^1[s,t]$. By a diagonal process, there exists $\gamma \in C^1(-\infty, t]$ such that $\gamma_n \to \gamma$ for the C^1-convergence on any compact of $(-\infty, t]$. By Lemma 21, γ is a one-sided minimizer with $\gamma(t) = x$. ■

4.1.2 Intersection of minimizers

The following lemma is a classical fact for one-sided minimizers.

Lemma 23. *Two different one-sided minimizers* $\gamma_1 \in C^1(-\infty, t_1]$ *and* $\gamma_2 \in C^1(-\infty, t_2]$ *cannot intersect each other more than once.*

Proof: suppose $\gamma_1(t) = \gamma_2(t)$ for $t \in \{t_3, t_4\}$, $t_3 < t_4$. Assume without loss of generality $\mathscr{A}_{t_3,t_4}(\gamma_1) \le \mathscr{A}_{t_3,t_4}(\gamma_2)$. Define the curve

$$\xi(s) = \gamma_2 \cdot \gamma_1 \cdot \gamma_2(s) = \begin{cases} \gamma_2(s), & s \in (-\infty, t_3] \cup [t_4, t_2], \\ \gamma_1(s), & s \in [t_3, t_4]. \end{cases}$$

Then, for $s < t_3$, one has $\mathscr{A}_{s,t_2}(\xi) \le \mathscr{A}_{s,t_2}(\gamma_2)$. Since $\dot\gamma_1(t_3) \neq \dot\gamma_2(t_3)$, ξ has a corner at t_3. By smoothing ξ, we obtain (by Prop. 18) a curve $\xi^* \in C^1(-\infty, t_2]$ such that $\xi^*(t_2) = \gamma_2(t_2)$ and

$$\mathscr{A}_{s,t_2}(\xi^*) < \mathscr{A}_{s,t_2}(\gamma_2)$$

for some $s < t_3$. This contradicts the fact that γ_2 is a one-sided minimizer. ■

The following lemma is also classical:

Lemma 24. *For all* $\eta > 0$*,* $T > 0$*, for a.e.* ω*, there exists a sequence of random times* $t_n(\omega) \to -\infty$ *such that*

$$\forall n, \max_{t_n - T \le s \le t_n} \sum_{k\ge1} \|F_k\|_{C^2(\mathbb{T}^1)} |\beta_k(s) - \beta_k(t_n)| \le \eta.$$

Proof: For $m \in \mathbb{N}$, consider the event

$$A_m = \left\{ \max_{(-m-1)T \leq s \leq -mT} \sum_{k \geq 1} \|F_k\|_{C^2(\mathbb{T}^1)} |\beta_k(s) - \beta_k(-mT)| \leq \eta \right\}.$$

The events A_m are independent and $\mathbb{P}(A_m) > 0$ for every m, hence, by the second Borel-Cantelli lemma,

$$\mathbb{P}(\{\omega \in A_m \text{ for infinitely many } m\}) = 1.$$

This gives the result for $t_n = -m_n T$. ∎

Lemma 24 asserts that, with probability one, the noise is arbitrary small on an infinite number of arbitrary long intervals. We use this result to prove the following

Lemma 25. *For a.e. ω, for all $\varepsilon > 0$, for any one-sided minimizers $\gamma_i \in C^1(-\infty, t_1]$, $i = 1, 2$, there exists $T = T(\varepsilon) > 0$ and an infinite sequence of random times $t_n(\omega, \varepsilon) \to -\infty$ such that, for all n, there exists some C^1-curve $\gamma_{1,2}$ (resp. $\gamma_{2,1}$) reconnecting γ_1 to γ_2 (resp. γ_2 to γ_1) between $t_n - T$ and t_n such that*

$$|\mathscr{A}_{t_n - T, t_n}(\gamma_i) - \mathscr{A}_{t_n - T, t_n}(\xi)| < \varepsilon, \quad i = 1, 2, \xi \in \{\gamma_{1,2}, \gamma_{2,1}\}.$$

We say that a C^1-curve $\gamma_{1,2}$ reconnects γ_1 to γ_2 between τ and t, $\tau < t$, if $\gamma_{1,2}$ and $\dot{\gamma}_{1,2}$ coincides with γ_1 and $\dot{\gamma}_1$ (*resp.* γ_2 and $\dot{\gamma}_2$) at τ (*resp.* t).

Proof of Lemma 25: fix $\eta = \eta(\varepsilon) > 0$, $T = T(\varepsilon) > 0$ and apply Lemma 24: there is a sequence $t_n(\omega, \varepsilon)$ such that

$$\forall n, \max_{t_n - T \leq s \leq t_n} \sum_{k \geq 1} \|F_k\|_{C^2(\mathbb{T}^1)} |\beta_k(s) - \beta_k(t_n)| \leq \eta.$$

Taking $T = (4\eta)^{-1}$ and repeating the proofs of Lemma 19 and Corollary 20, we obtain

$$\max_{t_n - T \leq s \leq t_n} |\dot{\gamma}_i(s)| \leq \frac{c}{T}, \quad i = 1, 2,$$

where c is a given numerical constant. Then it is easy to construct $\gamma_{1,2}$ and $\gamma_{2,1}$ with

$$|\dot{\gamma}_{1,2}(s)|, |\dot{\gamma}_{2,1}(s)| \leq \frac{c+1}{T}, \quad s \in [t_n - T, t_n].$$

We then compute $|\mathscr{A}_{t_n - T, t_n}(\gamma_i) - \mathscr{A}_{t_n - T, t_n}(\xi)| < \frac{c'}{T}$, where c' is a numerical constant, $i = 1, 2$, $\xi = \gamma_{1,2}$ or $\gamma_{2,1}$. The result follows by taking $T = \frac{c'}{\varepsilon}$. ∎

Lemma 25 contains the information that (as a result of the randomness of the force), two one-sided minimizers have an effective intersection at $-\infty$. As a result, in light of Lemma 23, we have the following

Theorem 26. *For a.e. ω, for any distinct C^1-curves γ_1 and γ_2 one-sided minimizers over $(-\infty, t_1]$ and $(-\infty, t_2]$ respectively, we have the following result. Assume that γ_1 and γ_2 intersect at a point (x, t). Then $t_1 = t_2 = t$ and $\gamma_1(t_1) = \gamma_2(t_2) = x$.*

Proof: suppose $t < t_1$ for example. The curve

$$\gamma_2 \cdot \gamma_1(s) := \begin{cases} \gamma_2(s), & s \in (-\infty, t], \\ \gamma_1(s), & s \in [t, t_1], \end{cases}$$

has a corner at time t. Hence, for a given $\tau < t$, it can be smoothed out according to Prop. 18 and the resulting curve $\gamma_2 \circ \gamma_1$ satisfies

$$\mathscr{A}_{\tau,t_1}(\gamma_2 \cdot \gamma_1) - \mathscr{A}_{\tau,t_1}(\gamma_2 \circ \gamma_1) = \delta > 0.$$

Set $\varepsilon = \delta/4$. Let (t_n) be given by Lemma 25 and take n large enough so that $\gamma_2 \circ \gamma_1 = \gamma_2$ on $(-\infty, t_n]$. We have

$$|\mathscr{A}_{t_n,t}(\gamma_2) - \mathscr{A}_{t_n,t}(\gamma_1)| \leq 2\varepsilon. \tag{37}$$

Indeed, in the case $\mathscr{A}_{t_n,t}(\gamma_2) - \mathscr{A}_{t_n,t}(\gamma_1) > 2\varepsilon$ for example, the curve

$$\gamma_2 \cdot \gamma_{2,1} \cdot \gamma_1(s) = \begin{cases} \gamma_2(s), & s \in (-\infty, t_n - T], \\ \gamma_{2,1}(s), & s \in [t_n - T, t_n], \\ \gamma_1(s), & s \in [t_n, t], \end{cases}$$

is a local perturbation of γ_2 with the same end at t and, for $s \leq t_n - T$,

$$\begin{aligned} \mathscr{A}_{s,t}(\gamma_2) - \mathscr{A}_{s,t}(\gamma_2 \cdot \gamma_{2,1} \cdot \gamma_1) &= \mathscr{A}_{t_n-T,t_n}(\gamma_2) - \mathscr{A}_{t_n-T,t_n}(\gamma_{2,1}) \\ &\quad + \mathscr{A}_{t_n,t}(\gamma_2) - \mathscr{A}_{t_n,t}(\gamma_1) \\ &\geq \varepsilon, \end{aligned}$$

which contradicts the fact that γ_2 is a one-sided minimizers. Now, the curve

$$\xi(s) = \gamma_1 \cdot \gamma_{1,2} \cdot \gamma_2 \circ \gamma_1(s) = \begin{cases} \gamma_1(s), & s \in (-\infty, t_n - T], \\ \gamma_{1,2}(s), & s \in [t_n - T, t_n], \\ \gamma_2 \circ \gamma_1(s), & s \in [t_n, t_1], \end{cases}$$

is a local perturbation of γ_1 with the same end at t_1 and, for $s \leq t_n - T$,

$$\begin{aligned} \mathscr{A}_{s,t_1}(\gamma_1) - \mathscr{A}_{s,t_1}(\xi) &= \mathscr{A}_{t_n-T,t_n}(\gamma_1) - \mathscr{A}_{t_n-T,t_n}(\gamma_{1,2}) \\ &\quad + \mathscr{A}_{t_n,t}(\gamma_1) - \mathscr{A}_{t_n,t}(\gamma_2) \\ &\quad + \mathscr{A}_{\tau,t_1}(\gamma_2 \cdot \gamma_1) - \mathscr{A}_{\tau,t_1}(\gamma_2 \circ \gamma_1) \\ &\geq \delta - 3\varepsilon > 0, \end{aligned}$$

and this contradicts the fact that γ_1 is a one-sided minimizer. ■

4.1.3 Invariant solution

We define now

$$\begin{aligned} u_+^\omega(x,t) &= \inf_{M(x,t,\omega)} \dot\gamma(t), \\ u_-^\omega(x,t) &= \sup_{M(x,t,\omega)} \dot\gamma(t), \end{aligned}$$

where the inf (*resp.* sup) is taken over the set $M(x,t,\omega)$ of one-sided minimizers γ such that $\gamma(t)=x$. The invariant solution is defined by $u=u_+$. Using the analysis in Section 4.1.1, 4.1.2 above, we will prove that for all t, for a.e. ω, $u^\omega(t)\in D$ and that the map $\Omega\to D$, $\omega\mapsto u^\omega(t)$ is measurable.

Lemma 27. *For a.e. ω, for every $t\in\mathbb{R}$, the set of $x\in\mathbb{T}^1$ with $\#M(x,t,\omega)>1$ is at most countable.*

Proof: To any $x\in\mathbb{T}^1$ with $\#M(x,t,\omega)>1$, and by Lemma 23, there corresponds a non-trivial segment $I(x)=[\gamma_1(t-1),\gamma_2(t-1)]$, where $\gamma_1<\gamma_2$ on $(-\infty,t)$. By Theorem 26, the segments $\{I(x)\}_{\{\#M(x,t,\omega)>1\}}$ are mutually disjoint. This gives the result. ■

It follows from Lemma 27 that, for a.e. ω, for every $t\in\mathbb{R}$, $u_+^\omega(x,t)=u_-^\omega(x,t)$, except possibly at a countable number of points $x\in\mathbb{T}^1$. Besides, we have

Lemma 28. *For a.e. ω, for every $t\in\mathbb{R}$, $x\in\mathbb{T}^1$,*

$$\begin{aligned} \lim_{y\uparrow x} u_+^\omega(y,t) &= u_-^\omega(x,t), \\ \lim_{y\downarrow x} u_+^\omega(y,t) &= u_+^\omega(x,t), \end{aligned}$$

and hence $u^\omega(t)\in D$.

Proof: We have $|u_+^\omega(y,t)|\le C(\omega,t)$, $y\in\mathbb{T}^1$, by Lemma 19. Assume that there exists $(y_n)\uparrow x$ with $u_+^\omega(y_n,t)\to v_*\in\mathbb{R}$. Pick up $\gamma_n\in M(y_n,t,\omega)$ such that $u_+^\omega(y_n,t)\le\dot\gamma_n(t)$. By repeating the arguments of Lemma 21 and Theorem 22, we see that, up to extraction of a subsequence, there is a one-sided minimizer $\gamma^*\in M(x,t,\omega)$ such that $\gamma_n\to\gamma^*$ for the C^1-convergence on any compact subset of $(-\infty,t]$. In particular $v_*=\dot\gamma^*(t)\le u_-^\omega(x,t)$. If $v_*=\dot\gamma^*(t)<u_-^\omega(x,t)$, then there exists $\gamma_*\in M(x,t,\omega)$ with $\dot\gamma_*(t)>\dot\gamma^*(t)$. But then, for n large enough, γ_n must intersect γ_*, which contradicts Theorem 26. ■

We now turn to the question of measurability of u. The space D is endowed with the Skohorod topology, and we denote by $\mathcal{B}(D)$ the σ-algebra of Borel sets. Any map $v\in D$ is redefined as a right-continuous function. If E is a dense subset of $\mathbb{T}^1$, then the cylinders

$$\pi_I^{-1}(A), \quad A\in\mathcal{B}(\mathbb{R}), I=\{x_1,\dots,x_k\}\subset E^k, \pi_I(v)=(v(x_1),\dots,v(x_k)),$$

generate $\mathcal{B}(D)$. In particular, if $v\colon \omega \mapsto v^\omega$ is a mapping $\Omega \to D$ such that $(\omega, x) \mapsto v^\omega(x)$ is measurable as a mapping $\Omega \times \mathbb{T}^1 \to \mathbb{R}$, then v is measurable $(\Omega, \mathcal{F}) \to (D, \mathcal{B}(D))$. We refer to Billingsley [Bil99] on this subject. The result is the following:

Lemma 29. *For all $t \in \mathbb{R}$, the mapping $(\Omega, \mathcal{F}) \to (D, \mathcal{B}(D))$, $\omega \mapsto u_+^\omega(\cdot, t)$ is measurable.*

Proof: Without loss of generality, suppose $t = 0$. Let (t_n) be a sequence of negative times such that $t_n \to -\infty$. Introduce, for $x \in \mathbb{T}^1$, the infimum of the action

$$A_n(x) := \inf_{\xi(0)=x} \mathscr{A}_{t_n,0}(\xi).$$

For all n, denote by $u_{n,+}^\omega$ the right-continuous solution to (31) starting from 0 at $t = t_n$. By Theorem 14 and Theorem 15, $u_{n,+}\colon \Omega \to D$ is measurable. Set

$$J = \{(\omega, x) \in \Omega \times \mathbb{T}^1; u_+^\omega(x, 0) \neq u_-^\omega(x, 0)\}.$$

Admit for the moment that J is measurable. Lemma 27 implies that for a.e. ω, the section $J^\omega = \{x \in \mathbb{T}^1, (\omega, x) \in J\}$ has Lebesgue measure 0. By Fubini's theorem, it follows that for $x \in E$, a set of full Lebesgue measure in $\mathbb{T}^1$, we have $\mathbb{P}(J_x) = 0$, where J_x is the section $\{\omega \in \Omega; (\omega, x) \in J\}$. For $x \in E$, we have $u_+^\omega(x, 0) = \lim_{n\to+\infty} u_{n,+}^\omega(x, 0)$ for a.e. ω; in particular $\omega \to u_+^\omega(x, 0)$ is measurable $(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Since E is dense in $\mathbb{T}^1$, $u_+(\cdot, 0)$ is measurable $(\Omega, \mathcal{F}) \to (D, \mathcal{B}(D)$.

To prove that J is measurable, consider the solution $\xi_{x,v}$ to the stochastic differential equation

$$d\xi(s) = \nu(s)ds, \quad d\nu(s) = \sum_{k\geq 1} f_k(\xi(s))d\beta_k(s), \quad s \leq 0$$

with data $\xi(0) = x$, $\dot{\xi}(0) = v$ (such a solution exists globally in time $s \leq 0$ since $\mathbb{T}^1$ is compact) and introduce the sets

$$\begin{aligned} E_n(k, m) = \{(\omega, x, v) \in \Omega \times \mathbb{T}^1 \times \mathbb{R};, |\dot{\xi}_{x,v}(0) - u_{n,+}^\omega(x, 0)| > \frac{1}{k}, \\ \mathscr{A}_{t_n,0}(\xi_{x,v}) - A_n(x) \leq \frac{1}{m}\}, \end{aligned}$$

Then each set $E_n(k, m)$ is measurable and so is

$$\tilde{J} := \mathrm{pr}_{1,2} \cup_{k,K\geq 1} \cap_{l,m\geq 1} (\cap_{p\geq 1} \cup_{n\geq p} E_n(k, l, m)),$$

where $\mathrm{pr}_{1,2}\colon \Omega \times \mathbb{T}^1 \times \mathbb{R} \to \Omega \times \mathbb{T}^1 \times$ is the projection $(\omega, x, v) \mapsto (\omega, x)$. If $(\omega, x) \in \tilde{J}$, then in particular there exists $v \in \mathbb{R}$ and a C^1-curve ξ such that $\xi(0) = 0$ and, for all $\varepsilon > 0$, $\mathscr{A}_{t_n,0}(\xi) - A_n(x) < \varepsilon$ for infinitely many n. This implies that ξ is a one-side minimizer (if $\gamma \in C^1(-\infty, 0]$ is a local perturbation of ξ which decreases the action $\mathscr{A}_{\tau,0}$ by $\delta > 0$ then we get a contradiction by taking $\varepsilon = \delta/2$ and $t_n \leq \tau$). Since $|\dot{\xi}_{x,v}(0) - u^\omega_{n,+}(x,0)| > \frac{1}{k}$ for a $k > 0$ and since any subsequence of $(u^\omega_{n,+}(x,0))$ is the slope at $t = 0$ of a one-sided minimizer taking the value x at $t = 0$, $\tilde{J}$ is the set of $(\omega, x) \in \Omega \times \mathbb{T}^1$ such that at least two one-sided minimizers arrive at $(x, 0)$, *i.e.* J. This concludes the proof of the lemma. ■

Proof of Theorem 16: For $c \in \mathbb{R}$, let

$$D_c = \{v \in D; \int_{\mathbb{T}^1} v = c\}.$$

Since $W = \partial_x \mathcal{W}$ by hypothesis, Eq. 31 preserves the "mass" $\int_{\mathbb{T}^1} u(\cdot, t)$, hence the evolution takes place in a D_c, $c \in \mathbb{R}$. We consider here the case $c = 0$ (since we constructed a solution starting from 0), but the case of a general c can be handled as well by a suitable modification of the action [EKMS00]. Therefore, we aim at proving the following statement (a little bit more precise than the statement of Theorem 16):

The stochastic inviscid Burgers' Equation (31) admits an invariant measure over D_0.

To that purpose, notice that, for $u = u_+$, we have $u^\omega(x,t) = u^{\theta^t\omega}(x,0)$ where $\theta^t\omega(s) := \omega(s+t)$ (the probability space Ω is taken to be the space of sequences $(\beta_k(\cdot))$ of $C(-\infty,\infty)$ with the Wiener measure), hence the law of $u(t)$ is the law of $u(0)$ and u is indeed invariant. By Prop. 9, there exists an invariant measure (the law of u) on D_0. ■

We emphasize the fact that, additionally, the invariant measure on D_0 is *unique*. We will not develop this point here, nor the properties of the invariant measure, see [EKMS00] on this subject.

Acknowledgment: I address all my thanks to Etienne Emmrich and Petra Wittbold, organizers of the Spring School *Analytical and Numerical Aspects of Evolution Equations* T.U. Berlin, 30/03/09–03/04/09.

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