The Theories of Nonlinear Diffusion

Juan Luis Vázquez

Departamento de Matemáticas
Universidad Autónoma de Madrid

2nd Spring School
Analytical and Numerical Aspects of Evolution

March 28 - April 1, 2010 at TU, Berlin
Outline

1. Theories of Diffusion
   - Diffusion
   - Heat equation
   - Linear Parabolic Equations
   - Nonlinear equations

2. Degenerate Diffusion
   - Introduction
   - The basics
   - Planning the theory
   - Asymptotic behaviour
   - References

3. Fast Diffusion Equation
1 Theories of Diffusion
   - Diffusion
   - Heat equation
   - Linear Parabolic Equations
   - Nonlinear equations

2 Degenerate Diffusion
   - Introduction
   - The basics
   - Planning the theory
   - Asymptotic behaviour
   - References

3 Fast Diffusion Equation
Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- what is diffusion anyway?

- how to explain it with mathematics?

- A main question is: how much of it can be explained with linear models, how much is essentially nonlinear?

- The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...

The Laplacian $\Delta$ is the King of Differential Operators.
Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- what is diffusion anyway?
- how to explain it with mathematics?
- A main question is: how much of it can be explained with linear models, how much is essentially nonlinear?

The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...

The Laplacian $\Delta$ is the King of Differential Operators.
Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- what is diffusion anyway?
- how to explain it with mathematics?
- A main question is: how much of it can be explained with linear models, how much is essentially nonlinear?
- The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...

The Laplacian $\Delta$ is the King of Differential Operators.
Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

- **what is diffusion anyway?**

- **how to explain it with mathematics?**

- **A main question is:** how much of it can be explained with *linear models*, how much is *essentially nonlinear*?

- The stationary states of diffusion belong to an important world, *elliptic equations*. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...

The Laplacian $\Delta$ is the King of Differential Operators.
Diffusion

Populations diffuse, substances (like particles in a solvent) diffuse, heat propagates, electrons and ions diffuse, the momentum of a viscous (Newtonian) fluid diffuses (linearly), there is diffusion in the markets, ...

• what is diffusion anyway?

• how to explain it with mathematics?

• A main question is: how much of it can be explained with linear models, how much is essentially nonlinear?

• The stationary states of diffusion belong to an important world, elliptic equations. Elliptic equations, linear and nonlinear, have many relatives: diffusion, fluid mechanics, waves of all types, quantum mechanics, ...

The Laplacian $\Delta$ is the King of Differential Operators.
**Diffusion in Wikipedia**

- **Diffusion.** The spreading of any quantity that can be described by the diffusion equation or a random walk model (e.g. concentration, heat, momentum, ideas, price) can be called diffusion.

- Some of the most important examples are listed below.
  * Atomic diffusion
  * Brownian motion, for example of a single particle in a solvent
  * Collective diffusion, the diffusion of a large number of (possibly interacting) particles
  * Effusion of a gas through small holes.
  * Electron diffusion, resulting in electric current
  * Facilitated diffusion, present in some organisms.
  * Gaseous diffusion, used for isotope separation
  * Heat flow
  * Ito- diffusion
  * Knudsen diffusion
  * Momentum diffusion, ex. the diffusion of the hydrodynamic velocity field
  * Osmosis is the diffusion of water through a cell membrane.
  * Photon diffusion
  * Reverse diffusion
  * Self-diffusion
  * Surface diffusion
Diffusion. The spreading of any quantity that can be described by the diffusion equation or a random walk model (e.g. concentration, heat, momentum, ideas, price) can be called diffusion.

Some of the most important examples are listed below.

- Atomic diffusion
- Brownian motion, for example of a single particle in a solvent
- Collective diffusion, the diffusion of a large number of (possibly interacting) particles
- Effusion of a gas through small holes.
- Electron diffusion, resulting in electric current
- Facilitated diffusion, present in some organisms.
- Gaseous diffusion, used for isotope separation
- Heat flow
- Ito-diffusion
- Knudsen diffusion
- Momentum diffusion, ex. the diffusion of the hydrodynamic velocity field
- Osmosis is the diffusion of water through a cell membrane.
- Photon diffusion
- Reverse diffusion
- Self-diffusion
- Surface diffusion
The heat equation origins

- We begin our presentation with the Heat Equation \( u_t = \Delta u \) and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.

They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on and from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

\[
 u(x, t) = \sum T_i(t)X_i(x)
\]

where the \( X_i(x) \) form the spectral sequence

\[
 -\Delta X_i = \lambda_i X_i.
\]

This is the famous linear eigenvalue problem, Spectral Theory.
The heat equation origins

We begin our presentation with the Heat Equation \( u_t = \Delta u \) and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application. They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on and from Physics.

The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

\[
    u(x, t) = \sum T_i(t)X_i(x)
\]

where the \( X_i(x) \) form the spectral sequence

\[
    -\Delta X_i = \lambda_i X_i.
\]

This is the famous linear eigenvalue problem, Spectral Theory.
The heat equation origins

- We begin our presentation with the Heat Equation \[ u_t = \Delta u \] and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application.

  They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on and from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

  \[ u(x, t) = \sum T_i(t)X_i(x) \]

  where the \( X_i(x) \) form the spectral sequence

  \[ -\Delta X_i = \lambda_i X_i. \]

  This is the famous linear eigenvalue problem, Spectral Theory.
The heat equation origins

- We begin our presentation with the Heat Equation \( u_t = \Delta u \) and the analysis proposed by Fourier, 1807, 1822 (Fourier decomposition, spectrum). The mathematical models of heat propagation and diffusion have made great progress both in theory and application. They have had a strong influence on 5 areas of Mathematics: PDEs, Functional Analysis, Inf. Dim. Dyn. Systems, Diff. Geometry and Probability. And on and from Physics.

- The heat flow analysis is based on two main techniques: integral representation (convolution with a Gaussian kernel) and mode separation:

\[
 u(x, t) = \sum T_i(t)X_i(x)
\]

where the \( X_i(x) \) form the spectral sequence

\[
 -\Delta X_i = \lambda_i X_i.
\]

This is the famous linear eigenvalue problem, Spectral Theory.
Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
⇒ Theory of Parabolic Equations

\[ u_t = \sum a_{ij} \partial_i \partial_j u + \sum b_i \partial_i u + cu + f \]

Main inventions in Parabolic Theory:
(1) \(a_{ij}, b_i, c, f\) regular ⇒ Maximum Principles, Schauder estimates, Harnack inequalities; \(C^\alpha\) spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded ⇒ \(W^{2,p}\) estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski, Wiener, Levy, Ito,...

\[ dX = \mu \, dt + \sigma \, dW \]
Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
⇒ Theory of Parabolic Equations

\[ u_t = \sum a_{ij} \partial_i \partial_j u + \sum b_i \partial_i u + cu + f \]

Main inventions in Parabolic Theory:
(1) \( a_{ij}, b_i, c, f \) regular ⇒ Maximum Principles, Schauder estimates,
Harnack inequalities; \( C^\alpha \) spaces (Hölder); potential theory; generation of
semigroups.
(2) coefficients only continuous or bounded ⇒ \( W^{2,p} \) estimates,
Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier,
Einstein, Smoluchowski, Wiener, Levy, Ito,...

\[ dX = \mu \, dt + \sigma \, dW \]
Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
⇒ Theory of Parabolic Equations

\[
\begin{align*}
    u_t &= \sum a_{ij} \partial_i \partial_j u + \sum b_i \partial_i u + cu + f
\end{align*}
\]

Main inventions in Parabolic Theory:
(1) \(a_{ij}, b_i, c, f\) regular ⇒ Maximum Principles, Schauder estimates,
Harnack inequalities; \(C^\alpha\) spaces (Hölder); potential theory; generation of
semigroups.
(2) coefficients only continuous or bounded ⇒ \(W^{2,p}\) estimates,
Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier,
Einstein, Smoluchowski, Wiener, Levy, Ito,...

\[
\begin{align*}
    dX &= \mu \, dt + \sigma \, dW
\end{align*}
\]
Linear heat flows

From 1822 until 1950 the heat equation has motivated
(i) Fourier analysis decomposition of functions (and set theory),
(ii) development of other linear equations
⇒ Theory of Parabolic Equations

\[ u_t = \sum a_{ij} \partial_i \partial_j u + \sum b_i \partial_i u + cu + f \]

Main inventions in Parabolic Theory:
(1) \( a_{ij}, b_i, c, f \) regular ⇒ Maximum Principles, Schauder estimates, Harnack inequalities; \( C^\alpha \) spaces (Hölder); potential theory; generation of semigroups.
(2) coefficients only continuous or bounded ⇒ \( W^{2,p} \) estimates, Calderón-Zygmund theory, weak solutions; Sobolev spaces.

The probabilistic approach: Diffusion as an stochastic process: Bachelier, Einstein, Smoluchowski, Wiener, Levy, Ito,...

\[ dX = \mu \, dt + \sigma \, dW \]
Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
- My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
- I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

\[ u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u) \]

- Typical reaction diffusion: Fujita model \( u_t = \Delta u + u^p \).
- The fluid flow models: Navier-Stokes or Euler equation.
- The geometrical models. Ricci and Yamabe flow
- The nonlinear wave and Shrödinger models: \( \partial_{tt} u = \Delta u + \Phi(u) \), \( i\partial_t u = -\Delta u + V(x)u + f(u) \).
Nonlinear heat flows

In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic. My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.

I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

\[ u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u) \]


Typical reaction diffusion: Fujita model \( u_t = \Delta u + u^p \).

The fluid flow models: Navier-Stokes or Euler equation.

The geometrical models. Ricci and Yamabe flow

The nonlinear wave and Shrödinger models: \( \partial_{tt} u = \Delta u + \Phi(u) \), \( i \partial_t u = -\Delta u + V(x)u + f(u) \).
Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
- My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
- I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

\[ u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u) \]


- Typical reaction diffusion: Fujita model \( u_t = \Delta u + u^p \).

- The fluid flow models: Navier-Stokes or Euler equation.

- The geometrical models. Ricci and Yamabe flow

- The nonlinear wave and Shrödinger models: 
  \[ \partial_{tt} u = \Delta u + \Phi(u), \]
  \[ i\partial_t u = -\Delta u + V(x)u + f(u). \]
Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic. My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
- I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

\[ u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u) \]

- Typical reaction diffusion: Fujita model \( u_t = \Delta u + u^p \).
- The fluid flow models: Navier-Stokes or Euler equation.
- The geometrical models. Ricci and Yamabe flow
- The nonlinear wave and Shrödinger models: \( \partial_{tt} u = \Delta u + \Phi(u) \), \( i\partial_t u = -\Delta u + V(x)u + f(u) \).
Nonlinear heat flows

In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic. My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.

I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

$$u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u)$$


Typical reaction diffusion: Fujita model $u_t = \Delta u + u^p$.

The fluid flow models: Navier-Stokes or Euler equation.

The geometrical models. Ricci and Yamabe flow

The nonlinear wave and Shrödinger models: $\partial_{tt} u = \Delta u + \Phi(u)$, $i\partial_t u = -\Delta u + V(x)u + f(u)$. 
Nonlinear heat flows

- In the last 50 years emphasis has shifted towards the Nonlinear World. Maths more difficult, more complex, and more realistic.
- My group works in the areas of Nonlinear Diffusion and Reaction Diffusion.
- I will talk about the theory mathematically called Nonlinear Parabolic PDEs. General formula

\[ u_t = \sum \partial_i A_i(u, \nabla u) + \sum B(x, u, \nabla u) \]

- Typical reaction diffusion: Fujita model \( u_t = \Delta u + u^p \).
- The fluid flow models: Navier-Stokes or Euler equation.
- The geometrical models. Ricci and Yamabe flow
- The nonlinear wave and Shrödinger models: \( \partial_{tt} u = \Delta u + \Phi(u) \), \( i\partial_t u = -\Delta u + V(x)u + f(u) \).
The Nonlinear Diffusion Models

- **The Stefan Problem** (Lamé and Clapeyron, 1833; Stefan 1880)

  \[
  SE : \begin{cases}
  u_t = k_1 \Delta u & \text{for } u > 0, \\
  u_t = k_2 \Delta u & \text{for } u < 0.
  \end{cases}
  \]

  \[
  TC : \begin{cases}
  u = 0, \\
  \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2).
  \end{cases}
  \]

  Main feature: the **free boundary** or **moving boundary** where \( u = 0 \). \text{TC=Transmission conditions at } u = 0.

- **The Hele-Shaw cell** (Hele-Shaw, 1898; Saffman-Taylor, 1958)

  \[
  u > 0, \ \Delta u = 0 \ \text{in} \ \Omega(t); \ u = 0, \ \mathbf{v} = L \partial_n u \ \text{on} \ \partial \Omega(t).
  \]

- **The Porous Medium Equation** \( \rightarrow \) *(hidden free boundary)*

  \[
  u_t = \Delta u^m, \quad m > 1.
  \]

  and other degenerate diffusion equations.

- **The \( p \)-Laplacian Equation**, 

  \[
  u_t = \text{div} \ (|\nabla u|^{p-2} \nabla u).
  \]

  and other gradient diffusivity equations.
The Nonlinear Diffusion Models

- **The Stefan Problem** (Lamé and Clapeyron, 1833; Stefan 1880)
  \[
  SE: \begin{cases} 
  u_t = k_1 \Delta u & \text{for } u > 0, \\
  u_t = k_2 \Delta u & \text{for } u < 0.
  \end{cases}
  \]
  TC: \[
  v = L(k_1 \nabla u_1 - k_2 \nabla u_2).
  \]

Main feature: the free boundary or moving boundary where \( u = 0 \). TC= Transmission conditions at \( u = 0 \).

- **The Hele-Shaw cell** (Hele-Shaw, 1898; Saffman-Taylor, 1958)
  \[
  u > 0, \ \Delta u = 0 \ \text{in } \Omega(t); \quad u = 0, \ v = L \partial_n u \ \text{on } \partial \Omega(t).
  \]

- **The Porous Medium Equation** \( \rightarrow (\text{hidden free boundary}) \)
  \[
  u_t = \Delta u^m, \quad m > 1.
  \]
  and other degenerate diffusion equations.

- **The \( p \)-Laplacian Equation**, \[
  u_t = \text{div} (|\nabla u|^{p-2} \nabla u).
  \]
  and other gradient diffusivity equations.
The Nonlinear Diffusion Models

- **The Stefan Problem** (Lamé and Clapeyron, 1833; Stefan 1880)

  \[ SE : \begin{cases} 
  u_t = k_1 \Delta u \quad &\text{for } u > 0, \\
  u_t = k_2 \Delta u \quad &\text{for } u < 0. 
  \end{cases} \]

  \[ TC : \begin{cases} 
  u = 0, \\
  \mathbf{v} = L(k_1 \nabla u_1 - k_2 \nabla u_2). 
  \end{cases} \]

  Main feature: the free boundary or moving boundary where \( u = 0 \). TC = Transmission conditions at \( u = 0 \).

- **The Hele-Shaw cell** (Hele-Shaw, 1898; Saffman-Taylor, 1958)

  \[ u > 0, \; \Delta u = 0 \; \text{in} \; \Omega(t); \; u = 0, \; \mathbf{v} = L \partial_n u \; \text{on} \; \partial \Omega(t). \]

- **The Porous Medium Equation** →*(hidden free boundary)*

  \[ u_t = \Delta u^m, \; m > 1. \]

  and other degenerate diffusion equations.

- **The \( p \)-Laplacian Equation,**

  \[ u_t = \text{div} \left( |\nabla u|^{p-2} \nabla u \right). \]

  and other gradient diffusivity equations.
The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)

\[ u_t = \Delta u + u^p \]

Main feature: If \( p > 1 \) the norm \( \| u(\cdot, t) \|_\infty \) of the solutions goes to infinity in finite time. Hint: Integrate \( u_t = u^p \).

Problem: what is the influence of diffusion / migration?

- General scalar model

\[ u_t = A(u) + f(u) \]

- The system model: \( \vec{\mathbf{u}} = (u_1, \cdots, u_m) \rightarrow \text{chemotaxis} \).

- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Any singularities? Other equations like Prandtl’s boundary layers.

- The geometrical models: the Ricci flow: \( \partial_t g_{ij} = -R_{ij} \). This is a nonlinear heat equation!
The Reaction Diffusion Models

- **The Standard Blow-Up model** (Kaplan, 1963; Fujita, 1966)
  \[ u_t = \Delta u + u^p \]

  Main feature: If \( p > 1 \) the norm \( \|u(\cdot, t)\|_\infty \) of the solutions goes to infinity in finite time. Hint: Integrate \( u_t = u^p \).
  
  **Problem:** what is the influence of diffusion / migration?

- **General scalar model**
  \[ u_t = A(u) + f(u) \]

- **The system model:** \( \vec{u} = (u_1, \cdots, u_m) \rightarrow \text{chemotaxis} \).

- **The fluid flow models:** Navier-Stokes or Euler equation systems for incompressible flow. **Any singularities?** Other equations like Prandtl’s boundary layers.

- **The geometrical models:** the Ricci flow: \( \partial_t g_{ij} = -R_{ij} \). This is a nonlinear heat equation!
The Reaction Diffusion Models

- The Standard Blow-Up model (Kaplan, 1963; Fujita, 1966)
  \[ u_t = \Delta u + u^p \]
  Main feature: If \( p > 1 \) the norm \( \|u(\cdot, t)\|_\infty \) of the solutions goes to infinity in finite time. Hint: Integrate \( u_t = u^p \).
  Problem: what is the influence of diffusion / migration?

- General scalar model
  \[ u_t = A(u) + f(u) \]

- The system model: \( \vec{u} = (u_1, \cdots, u_m) \rightarrow \text{chemotaxis} \).

- The fluid flow models: Navier-Stokes or Euler equation systems for incompressible flow. Any singularities? Other equations like Prandtl’s boundary layers.

- The geometrical models: the Ricci flow: \( \partial_t g_{ij} = -R_{ij} \). This is a nonlinear heat equation!
An opinion of John Nash, 1958:

The open problems in the area of nonlinear p.d.e. are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid,...

“Continuity of solutions of elliptic and parabolic equations”, paper published in Amer. J. Math, 80, no 4 (1958), 931-954
An opinion of John Nash, 1958:

The open problems in the area of nonlinear p.d.e. are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid,...

“Continuity of solutions of elliptic and parabolic equations”, paper published in Amer. J. Math, 80, no 4 (1958), 931-954
An opinion of John Nash, 1958:

The open problems in the area of nonlinear p.d.e. are very relevant to applied mathematics and science as a whole, perhaps more so that the open problems in any other area of mathematics, and the field seems poised for rapid development. It seems clear, however, that fresh methods must be employed...

Little is known about the existence, uniqueness and smoothness of solutions of the general equations of flow for a viscous, compressible, and heat conducting fluid,...

“Continuity of solutions of elliptic and parabolic equations”, paper published in Amer. J. Math, 80, no 4 (1958), 931-954
Outline

1 Theories of Diffusion
   - Diffusion
   - Heat equation
   - Linear Parabolic Equations
   - Nonlinear equations

2 Degenerate Diffusion
   - Introduction
   - The basics
   - Planning the theory
   - Asymptotic behaviour
   - References

3 Fast Diffusion Equation
The Porous Medium - Fast Diffusion Equation

If you go to Wikipedia and look for the Diffusion Equation you will find

\[
\frac{\partial \phi(\vec{r}, t)}{\partial t} = \nabla \cdot (D(\phi, \vec{r}) \nabla \phi(\vec{r}, t))
\]

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is

\[ u_t = \Delta u^m = \nabla \cdot (c(u) \nabla u) \]

\( c(u) \) indicates density-dependent diffusivity

\[ c(u) = mu^{m-1} = m|u|^{m-1} \]

- If \( m > 1 \) it degenerates at \( u = 0 \), \( \implies \) slow diffusion
- For \( m = 1 \) we get the classical Heat Equation.
- On the contrary, if \( m < 1 \) it is singular at \( u = 0 \) \( \implies \) Fast Diffusion.
- But power functions are tricky:
  - \( c(u) \to 0 \) as \( u \to \infty \) if \( m > 1 \) (“slow case”)
  - \( c(u) \to \infty \) as \( u \to \infty \) if \( m < 1 \) (“fast case”)

Juan Luis Vázquez (Univ. Autónoma de Madrid)
Theory of Nonlinear Diffusion
March 28 - April 1, 2010 at TU, Berlin
The Porous Medium - Fast Diffusion Equation

If you go to Wikipedia and look for the Diffusion Equation you will find

$$\frac{\partial \phi(\vec{r}, t)}{\partial t} = \nabla \cdot (D(\phi, \vec{r}) \nabla \phi(\vec{r}, t))$$

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is

$$u_t = \Delta u^m = \nabla \cdot (c(u) \nabla u)$$

c(u) indicates density-dependent diffusivity

$$c(u) = mu^{m-1} = m|u|^{m-1}$$

- If $m > 1$ it degenerates at $u = 0$, $\implies$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0 \implies$ Fast Diffusion.
- But power functions are tricky:
  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ (“slow case”)
  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ (“fast case”)
The Porous Medium - Fast Diffusion Equation

If you go to Wikipedia and look for the Diffusion Equation you will find

\[
\frac{\partial \phi(\vec{r}, t)}{\partial t} = \nabla \cdot (D(\phi, \vec{r}) \nabla \phi(\vec{r}, t))
\]

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is

\[
u_t = \Delta u^m = \nabla \cdot (c(u) \nabla u)
\]

\(c(u)\) indicates density-dependent diffusivity

\[
c(u) = mu^{m-1} = m|u|^{m-1}
\]

- If \(m > 1\) it degenerates at \(u = 0\), \(\implies\) slow diffusion
- For \(m = 1\) we get the classical Heat Equation.
- On the contrary, if \(m < 1\) it is singular at \(u = 0\) \(\implies\) Fast Diffusion.
- But power functions are tricky:
  - \(c(u) \to 0\) as \(u \to \infty\) if \(m > 1\) ("slow case")
  - \(c(u) \to \infty\) as \(u \to \infty\) if \(m < 1\) ("fast case")
The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

\[
\frac{\partial \phi(\vec{r}, t)}{\partial t} = \nabla \cdot (D(\phi, \vec{r}) \nabla \phi(\vec{r}, t))
\]

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is

\[
u_t = \Delta u^m = \nabla \cdot (c(u) \nabla u)
\]

\(c(u)\) indicates density-dependent diffusivity

\[
c(u) = mu^{m-1} [= m|u|^{m-1}]
\]

- If \(m > 1\) it degenerates at \(u = 0\), \(\implies\) slow diffusion
- For \(m = 1\) we get the classical Heat Equation.
- On the contrary, if \(m < 1\) it is singular at \(u = 0 \implies\) Fast Diffusion.
- But power functions are tricky:
  - \(c(u) \to 0\) as \(u \to \infty\) if \(m > 1\) (“slow case”)
  - \(c(u) \to \infty\) as \(u \to \infty\) if \(m < 1\) (“fast case”)

Juan Luis Vázquez (Univ. Autónoma de Madrid)  Theory of Nonlinear Diffusion  March 28 - April 1, 2010 at TU, Berlin 13 / 27
If you go to Wikipedia and look for the Diffusion Equation you will find

$$\frac{\partial \phi(\vec{r}, t)}{\partial t} = \nabla \cdot (D(\phi, \vec{r}) \nabla \phi(\vec{r}, t))$$

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is

$$u_t = \Delta u^m = \nabla \cdot (c(u) \nabla u)$$

$c(u)$ indicates density-dependent diffusivity

$$c(u) = mu^{m-1} = m|u|^{m-1}$$

- If $m > 1$ it degenerates at $u = 0$, $\Rightarrow$ slow diffusion
- For $m = 1$ we get the classical Heat Equation.
- On the contrary, if $m < 1$ it is singular at $u = 0$ $\Rightarrow$ Fast Diffusion.
- But power functions are tricky:
  - $c(u) \to 0$ as $u \to \infty$ if $m > 1$ ("slow case")
  - $c(u) \to \infty$ as $u \to \infty$ if $m < 1$ ("fast case")
The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

\[
\frac{\partial \phi(\vec{r}, t)}{\partial t} = \nabla \cdot (D(\phi, \vec{r}) \nabla \phi(\vec{r}, t))
\]

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is

\[
\begin{aligned}
  u_t &= \Delta u^m = \nabla \cdot (c(u)\nabla u) \\
  c(u) &= m|u|^{m-1}
\end{aligned}
\]

c(u) indicates density-dependent diffusivity

- If \( m > 1 \) it degenerates at \( u = 0 \), \( \Rightarrow \) slow diffusion
- For \( m = 1 \) we get the classical Heat Equation.
- On the contrary, if \( m < 1 \) it is singular at \( u = 0 \) \( \Rightarrow \) Fast Diffusion.

But power functions are tricky:
- \( c(u) \to 0 \) as \( u \to \infty \) if \( m > 1 \) (“slow case”)
- \( c(u) \to \infty \) as \( u \to \infty \) if \( m < 1 \) (“fast case”)
The Porous Medium - Fast Diffusion Equation

- If you go to Wikipedia and look for the Diffusion Equation you will find

\[
\frac{\partial \phi(\vec{r}, t)}{\partial t} = \nabla \cdot \left( D(\phi, \vec{r}) \nabla \phi(\vec{r}, t) \right)
\]

It is not difficult from here to conclude that the simplest model of nonlinear diffusion equation is

\[
u_t = \Delta u^m = \nabla \cdot (c(u) \nabla u)
\]

\(c(u)\) indicates density-dependent diffusivity

\[c(u) = mu^{m-1}[= m|u|^{m-1}]\]

- If \(m > 1\) it degenerates at \(u = 0\), \(\implies\) slow diffusion
- For \(m = 1\) we get the classical Heat Equation.
- On the contrary, if \(m < 1\) it is singular at \(u = 0\) \(\implies\) Fast Diffusion.
- But power functions are tricky:
  - \(c(u) \to 0\) as \(u \to \infty\) if \(m > 1\) (“slow case”)
  - \(c(u) \to \infty\) as \(u \to \infty\) if \(m < 1\) (“fast case”)

Juan Luis Vázquez (Univ. Autónoma de Madrid)  Theory of Nonlinear Diffusion  March 28 - April 1, 2010 at TU, Berlin  13 / 27
The basics

- For $m = 2$ the equation is re-written as

$$\frac{1}{2} u_t = u \Delta u + |\nabla u|^2$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$u_t = |\nabla u|^2$$

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$v_t = (m - 1) v \Delta v + |\nabla v|^2$$

where $v = cu^{m-1}$ is the pressure; normalization $c = m / (m - 1)$.

This separates $m > 1$ PME - from - $m < 1$ FDE.
The basics

For for $m = 2$ the equation is re-written as

$$\frac{1}{2} u_t = u\Delta u + |\nabla u|^2$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$u_t = |\nabla u|^2$$

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$v_t = (m - 1)v\Delta v + |\nabla v|^2$$

where $v = cu^{m-1}$ is the pressure; normalization $c = m/(m - 1)$. This separates $m > 1$ PME - from - $m < 1$ FDE.
The basics

For $m = 2$ the equation is re-written as

$$\frac{1}{2} u_t = u \Delta u + |\nabla u|^2$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$u_t = |\nabla u|^2$$

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$v_t = (m - 1) v \Delta v + |\nabla v|^2$$

where $v = cu^{m-1}$ is the pressure; normalization $c = m/(m - 1)$. This separates $m > 1$ PME - from - $m < 1$ FDE
The basics

- For $m = 2$ the equation is re-written as

\[ \frac{1}{2} u_t = u \Delta u + |\nabla u|^2 \]

and you can see that for $u \sim 0$ it looks like the eikonal equation

\[ u_t = |\nabla u|^2 \]

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

\[ v_t = (m - 1)v \Delta v + |\nabla v|^2 \]

where $v = cu^{m-1}$ is the pressure; normalization $c = m/(m - 1)$. This separates $m > 1$ PME - from - $m < 1$ FDE.
The basics

For for $m = 2$ the equation is re-written as

$$\frac{1}{2} u_t = u \Delta u + |\nabla u|^2$$

and you can see that for $u \sim 0$ it looks like the eikonal equation

$$u_t = |\nabla u|^2$$

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

No big problem when $m > 1$, $m \neq 2$. The pressure transformation gives:

$$v_t = (m - 1)v \Delta v + |\nabla v|^2$$

where $v = cu^{m-1}$ is the pressure; normalization $c = m/(m - 1)$. This separates $m > 1$ PME - from - $m < 1$ FDE
The basics

- For $m = 2$ the equation is re-written as
  \[ \frac{1}{2} u_t = u \Delta u + |\nabla u|^2 \]

and you can see that for $u \sim 0$ it looks like the eikonal equation
  \[ u_t = |\nabla u|^2 \]

This is not parabolic, but hyperbolic (propagation along characteristics). Mixed type, mixed properties.

- No big problem when $m > 1, m \neq 2$. The pressure transformation gives:
  \[ v_t = (m - 1) v \Delta v + |\nabla v|^2 \]

where $v = cu^{m-1}$ is the pressure; normalization $c = m/(m - 1)$.
This separates $m > 1$ PME - from - $m < 1$ FDE.
Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)
  \[ m = 1 + \gamma \geq 2 \]
  \[
  \begin{cases}
  \rho_t + \text{div} (\rho \mathbf{v}) = 0, \\
  \mathbf{v} = -\frac{k}{\mu} \nabla p, \quad p = p(\rho).
  \end{cases}
  \]

Second line left is the Darcy law for flows in porous media (Darcy, 1856). Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.

To the right, put \( p = p_0 \rho^\gamma \), with \( \gamma = 1 \) (isothermal), \( \gamma > 1 \) (adiabatic flow).

\[
\rho_t = \text{div} \left( \frac{k}{\mu} \rho \nabla p \right) = \text{div} \left( \frac{k}{\mu} \rho \nabla (p_0 \rho^\gamma) \right) = c\Delta \rho^{\gamma+1}.
\]

- Underground water infiltration (Boussinesq, 1903) \( m = 2 \)
Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)
  \[ m = 1 + \gamma \geq 2 \]
  \[
  \begin{align*}
  \rho_t + \text{div} (\rho \mathbf{v}) &= 0, \\
  \mathbf{v} &= -\frac{k}{\mu} \nabla p, \quad p = p(\rho).
  \end{align*}
  \]
  Second line left is the Darcy law for flows in porous media (Darcy, 1856). Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.

- To the right, put \( p = p_o \rho^\gamma \), with \( \gamma = 1 \) (isothermal), \( \gamma > 1 \) (adiabatic flow).

  \[
  \rho_t = \text{div} \left( \frac{k}{\mu} \rho \nabla p \right) = \text{div} \left( \frac{k}{\mu} \rho \nabla (p_o \rho^\gamma) \right) = c \Delta \rho^{\gamma+1}.
  \]

- Underground water infiltration (Boussinesq, 1903) \( m = 2 \)
Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)
  \[ m = 1 + \gamma \geq 2 \]

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{v}) &= 0, \\
\mathbf{v} &= -\frac{k}{\mu} \nabla p, \quad p = p(\rho).
\end{align*}
\]

Second line left is the Darcy law for flows in porous media (Darcy, 1856).

*Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.*

To the right, put \( p = p_o \rho^\gamma \), with \( \gamma = 1 \) (isothermal), \( \gamma > 1 \) (adiabatic flow).

\[ \rho_t = \text{div} \left( \frac{k}{\mu} \rho \nabla p \right) = \text{div} \left( \frac{k}{\mu} \rho \nabla (p_o \rho^\gamma) \right) = c \Delta \rho^{\gamma+1}. \]

- Underground water infiltration (Boussinesq, 1903) \( m = 2 \)
Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)

\[ m = 1 + \gamma \geq 2 \]

\[
\begin{align*}
\rho_t + \text{div} (\rho \mathbf{v}) &= 0, \\
\mathbf{v} &= -\frac{k}{\mu} \nabla p, \quad p = p(\rho).
\end{align*}
\]

Second line left is the Darcy law for flows in porous media (Darcy, 1856). Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.

To the right, put \( p = p_o \rho^\gamma \), with \( \gamma = 1 \) (isothermal), \( \gamma > 1 \) (adiabatic flow).

\[
\rho_t = \text{div} \left( \frac{k}{\mu} \rho \nabla p \right) = \text{div} \left( \frac{k}{\mu} \rho \nabla (p_o \rho^\gamma) \right) = c\Delta \rho^{\gamma+1}.
\]

- Underground water infiltration (Boussinesq, 1903) \( m = 2 \)
Applied motivation for the PME

- Flow of gas in a porous medium (Leibenzon, 1930; Muskat 1933)
  \[ m = 1 + \gamma \geq 2 \]

  \[
  \begin{aligned}
  \rho_t + \text{div}(\rho \mathbf{v}) &= 0, \\
  \mathbf{v} &= -\frac{k}{\mu} \nabla p, \quad p = p(\rho).
  \end{aligned}
  \]

  Second line left is the Darcy law for flows in porous media (Darcy, 1856).
  *Porous media flows are potential flows due to averaging of Navier-Stokes on the pore scales.*

  To the right, put \( p = p_o \rho^\gamma \), with \( \gamma = 1 \) (isothermal), \( \gamma > 1 \) (adiabatic flow).

  \[
  \rho_t = \text{div} \left( \frac{k}{\mu} \rho \nabla p \right) = \text{div} \left( \frac{k}{\mu} \rho \nabla (p_o \rho^\gamma) \right) = c \Delta \rho^{\gamma+1}.
  \]

- Underground water infiltration (Boussinesq, 1903) \( m = 2 \)
Applied motivation II

- Plasma radiation $m \geq 4$ (Zeldovich-Raizer, 1950)
  Experimental fact: diffusivity at high temperatures is not constant as in Fourier’s law, due to radiation.

$$\frac{d}{dt} \int_{\Omega} c \rho T \, dx = \int_{\partial \Omega} k(T) \nabla T \cdot n \, dS.$$ 

Put $k(T) = k_0 T^n$, apply Gauss law and you get

$$c \rho \frac{\partial T}{\partial t} = \text{div}(k(T) \nabla T) = c_1 \Delta T^{n+1}.$$ 

→ When $k$ is not a power we get $T_t = \Delta \Phi(T)$ with $\Phi'(T) = k(T)$.

- Spreading of populations (self-avoiding diffusion) $m \sim 2$.
- Thin films under gravity (no surface tension) $m = 4$.
- Kinetic limits (Carleman models, McKean, PL Lions and Toscani et al.)
- Many more (boundary layers, geometry).
Applied motivation II

- Plasma radiation $m \geq 4$ (Zeldovich-Raizer, 1950)
  Experimental fact: diffusivity at high temperatures is not constant as in Fourier’s law, due to radiation.

$$\frac{dT}{dt} \int_{\Omega} c_{\rho} T \, dx = \int_{\partial \Omega} k(T) \nabla T \cdot n \, dS.$$ 

Put $k(T) = k_0 T^n$, apply Gauss law and you get

$$c_{\rho} \frac{\partial T}{\partial t} = \text{div}(k(T) \nabla T) = c_1 \Delta T^{n+1}.$$ 

→ When $k$ is not a power we get $T_t = \Delta \Phi(T)$ with $\Phi'(T) = k(T)$.

- Spreading of populations (self-avoiding diffusion) $m \sim 2$.
- Thin films under gravity (no surface tension) $m = 4$.
- Kinetic limits (Carleman models, McKean, PL Lions and Toscani et al.)
- Many more (boundary layers, geometry).
Plasma radiation $m \geq 4$ (Zeldovich-Raizer, 1950)
Experimental fact: diffusivity at high temperatures is not constant as in Fourier’s law, due to radiation.

\[
\frac{d}{dt} \int_{\Omega} c\rho T\,dx = \int_{\partial\Omega} k(T)\nabla T \cdot \mathbf{n} dS.
\]

Put $k(T) = k_o T^n$, apply Gauss law and you get

\[
c\rho \frac{\partial T}{\partial t} = \text{div}(k(T)\nabla T) = c_1 \Delta T^{n+1}.
\]

→ When $k$ is not a power we get $T_t = \Delta \Phi(T)$ with $\Phi'(T) = k(T)$.

- Spreading of populations (self-avoiding diffusion) $m \sim 2$.
- Thin films under gravity (no surface tension) $m = 4$.
- Kinetic limits (Carleman models, McKean, PL Lions and Toscani et al.)
- Many more (boundary layers, geometry).
Applied motivation II

- Plasma radiation $m \geq 4$ (Zeldovich-Raizer, 1950)
  Experimental fact: diffusivity at high temperatures is not constant as in Fourier’s law, due to radiation.

$$\frac{d}{dt} \int_{\Omega} c_\rho T \, dx = \int_{\partial\Omega} k(T) \nabla T \cdot ndS.$$ 

Put $k(T) = k_0 T^n$, apply Gauss law and you get

$$c_\rho \frac{\partial T}{\partial t} = \text{div}(k(T) \nabla T) = c_1 \Delta T^{n+1}.$$ 

$\rightarrow$ When $k$ is not a power we get $T_t = \Delta \Phi(T)$ with $\Phi'(T) = k(T)$.

- Spreading of populations (self-avoiding diffusion) $m \sim 2$.
- Thin films under gravity (no surface tension) $m = 4$.
- Kinetic limits (Carleman models, McKean, PL Lions and Toscani et al.)

$\rightarrow$ Many more (boundary layers, geometry).


Applied motivation II

- Plasma radiation \( m \geq 4 \) (Zeldovich-Raizer, 1950)
  Experimental fact: diffusivity at high temperatures is not constant as in Fourier’s law, due to radiation.

\[
\frac{d}{dt} \int_{\Omega} c \rho T \, dx = \int_{\partial \Omega} k(T) \nabla T \cdot n \, dS.
\]

Put \( k(T) = k_0 T^n \), apply Gauss law and you get

\[
c \rho \frac{\partial T}{\partial t} = \text{div}(k(T) \nabla T) = c_1 \Delta T^{n+1}.
\]

→ When \( k \) is not a power we get \( T_t = \Delta \Phi(T) \) with \( \Phi'(T) = k(T) \).

- Spreading of populations (self-avoiding diffusion) \( m \sim 2 \).
- Thin films under gravity (no surface tension) \( m = 4 \).
- Kinetic limits (Carleman models, McKean, PL Lions and Toscani et al.)
- Many more (boundary layers, geometry).
Planning of the Theory

These are the main topics of mathematical analysis (1958-2000):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: is there a limit? $C^k$ for some $k$?
- Regularity and movement of interfaces: $C^k$ for some $k$.
- Asymptotic behaviour: patterns and rates? universal?
- The probabilistic approach. Nonlinear process. Wasserstein estimates
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Jacobi: A question of analysis is worth as much as a question on the system of the world.
Planning of the Theory

These are the main topics of mathematical analysis (1958-2000):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: is there a limit? $C^k$ for some $k$?
- Regularity and movement of interfaces: $C^k$ for some $k$?
- Asymptotic behaviour: patterns and rates? universal?
- The probabilistic approach. Nonlinear process. Wasserstein estimates
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Jacobi: A question of analysis is worth as much as a question on the system of the world.
Planning of the Theory

These are the main topics of mathematical analysis (1958-2000):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: is there a limit? $C^k$ for some $k$?
- Regularity and movement of interfaces: $C^k$ for some $k$.
- Asymptotic behaviour: patterns and rates? universal?
- The probabilistic approach. Nonlinear process. Wasserstein estimates
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Jacobi: A question of analysis is worth as much as a question on the system of the world.
Planning of the Theory

These are the main topics of mathematical analysis (1958-2000):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: is there a limit? $C^k$ for some $k$?
- Regularity and movement of interfaces: $C^k$ for some $k$?
- Asymptotic behaviour: patterns and rates? universal?
- The probabilistic approach. Nonlinear process. Wasserstein estimates
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Jacobi: A question of analysis is worth as much as a question on the system of the world.
Planning of the Theory

These are the main topics of mathematical analysis (1958-2000):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit? $C^k$ for some $k$?*
- Regularity and movement of interfaces: $C^k$ for some $k$?
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Jacobi: A question of analysis is worth as much as a question on the system of the world.
Planning of the Theory

These are the main topics of mathematical analysis (1958-2000):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: is there a limit? $C^k$ for some $k$?
- Regularity and movement of interfaces: $C^k$ for some $k$.
- Asymptotic behaviour: patterns and rates? universal?
- The probabilistic approach. Nonlinear process. Wasserstein estimates
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Jacobi: A question of analysis is worth as much as a question on the system of the world.
Planning of the Theory

These are the main topics of mathematical analysis (1958-2000):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: *is there a limit?* $C^k$ for some $k$?
- Regularity and movement of interfaces: $C^k$ for some $k$.
- Asymptotic behaviour: *patterns and rates? universal?*
- The probabilistic approach. *Nonlinear process. Wasserstein estimates*
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

*Jacobi: A question of analysis is worth as much as a question on the system of the world.*
Planning of the Theory

These are the main topics of mathematical analysis (1958-2000):

- The precise meaning of solution.
- The nonlinear approach: estimates; functional spaces.
- Existence, non-existence. Uniqueness, non-uniqueness.
- Regularity of solutions: is there a limit? $C^k$ for some $k$?
- Regularity and movement of interfaces: $C^k$ for some $k$?
- Asymptotic behaviour: patterns and rates? universal?
- The probabilistic approach. Nonlinear process. Wasserstein estimates
- Generalization: fast models, inhomogeneous media, anisotropic media, applications to geometry or image processing; other effects.

Jacobi: A question of analysis is worth as much as a question on the system of the world.
Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles.
  They are source solutions. **Source means that** $u(x, t) \to M \delta(x)$ **as** $t \to 0$.
- Explicit formulas (1950):
  \[
  B(x, t; M) = t^{-\alpha} F(x/t^\beta), \quad F(\xi) = \left(C - k\xi^2\right)^{1/(m-1)}
  \]
  \[
  \alpha = \frac{n}{2+n(m-1)}
  \]
  \[
  \beta = \frac{1}{2+n(m-1)} < 1/2
  \]
  Height $u = Ct^{-\alpha}$
  Free boundary at distance $|x| = ct^\beta$

Scaling law; anomalous diffusion versus Brownian motion
Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles. They are source solutions. *Source means that* \( u(x, t) \rightarrow M \delta(x) \) *as* \( t \rightarrow 0 \).
- Explicit formulas (1950):

\[
B(x, t; M) = t^{-\alpha} F(x/t^\beta), \quad F(\xi) = \left(C - k\xi^2\right)^{1/(m-1)}
\]

\[
\alpha = \frac{n}{2+n(m-1)}
\]

\[
\beta = \frac{1}{2+n(m-1)} < 1/2
\]

Height \( u = Ct^{-\alpha} \)

Free boundary at distance \( |x| = ct^\beta \)

Scaling law; anomalous diffusion versus Brownian motion
Barenblatt profiles (ZKB)

- These profiles are the alternative to the Gaussian profiles. They are source solutions. *Source means that* $u(x, t) \to M \delta(x)$ *as* $t \to 0$.
- Explicit formulas (1950):

  $$B(x, t; M) = t^{-\alpha} F(x/t^\beta), \quad F(\xi) = \left(C - k\xi^2\right)^{1/(m-1)}$$

  $$\alpha = \frac{n}{2+n(m-1)}$$

  $$\beta = \frac{1}{2+n(m-1)} < 1/2$$

  Height $u = Ct^{-\alpha}$

  Free boundary at distance $|x| = ct^\beta$

Scaling law; anomalous diffusion versus Brownian motion
Parabolic to Elliptic

- **Semigroup solution / mild solution.** The typical product of functional discretization schemes:  
  \[ u = \{u_n\}_n, \quad u_n = u(\cdot, t_n), \]

  \[ u_t = \Delta \Phi(u), \quad \frac{u_n - u_{n-1}}{h} - \Delta \Phi(u_n) = 0. \]

  Now put \( f := u_{n-1}, \ u := u_n, \) and \( v = \Phi(u), \ u = \beta(v): \)

  \[ -h\Delta \Phi(u) + u = f, \quad -h\Delta v + \beta(v) = f. \]

“Nonlinear elliptic equations”; Crandall-Liggett Theorems authors: Benilan, Brezis; Vazquez; Ambrosio, Savarè, Nochetto; Brezis, Ponce, Marcus

- **Separation of variables.** Put \( u(x, t) = F(x)G(t). \) Then PME gives

  \[ F(x)G'(t) = G^m(t)\Delta F^m(x), \]

  so that \( G'(t) = -G^m(t), \) i.e., \( G(t) = (m - 1)t^{-1/(m-1)} \) if \( m > 1 \) and

  \[ -\Delta F^m(x) = F(x), \quad -\Delta v(x) = v^p(x), \ p = 1/m. \]

  This is more interesting for \( m < 1, \) specially for \( m = (n - 2)/(n + 2). \)
Parabolic to Elliptic

- **Semigroup solution / mild solution.** The typical product of functional discretization schemes: \( u = \{u_n\}_n, u_n = u(\cdot, t_n), \)

\[
\begin{align*}
  u_t &= \Delta \Phi(u), \\
  \frac{u_n - u_{n-1}}{h} - \Delta \Phi(u_n) &= 0.
\end{align*}
\]

Now put \( f := u_{n-1}, u := u_n, \) and \( \nu = \Phi(u), \) \( u = \beta(\nu): \)

\[
- h \Delta \Phi(u) + u = f, \quad - h \Delta \nu + \beta(\nu) = f.
\]

"Nonlinear elliptic equations"; Crandall-Ligget Theorems authors: Benilan, Brezis; Vazquez; Ambrosio, Savarè, Nochetto; Brezis, Ponce, Marcus

- **Separation of variables.** Put \( u(x, t) = F(x)G(t). \) Then PME gives

\[
F(x)G'(t) = G^m(t)\Delta F^m(x),
\]

so that \( G'(t) = -G^m(t), \) i.e., \( G(t) = (m - 1)t^{-1/(m-1)} \) if \( m > 1 \) and

\[
-\Delta F^m(x) = F(x),
\]

\[
-\Delta \nu(x) = \nu^p(x), \quad p = 1/m.
\]

This is more interesting for \( m < 1, \) specially for \( m = (n-2)/(n+2). \)
Semigroup solution / mild solution. The typical product of functional discretization schemes: \( u = \{u_n\}_n, u_n = u(\cdot, t_n), \)

\[ u_t = \Delta \Phi(u), \quad \frac{u_n - u_{n-1}}{h} - \Delta \Phi(u_n) = 0. \]

Now put \( f := u_{n-1}, u := u_n, \) and \( v = \Phi(u), u = \beta(v): \)

\[ -h\Delta \Phi(u) + u = f, \quad -h\Delta v + \beta(v) = f. \]

"Nonlinear elliptic equations"; Crandall-Liggett Theorems authors: Benilan, Brezis; Vazquez; Ambrosio, Savarè, Nochetto; Brezis, Ponce, Marcus

Separation of variables. Put \( u(x, t) = F(x)G(t). \) Then PME gives

\[ F(x)G'(t) = G^m(t)\Delta F^m(x), \]

so that \( G'(t) = -G^m(t), \) i.e., \( G(t) = (m - 1)t^{-1/(m-1)} \) if \( m > 1 \) and

\[ -\Delta F^m(x) = F(x), \]

\[ -\Delta v(x) = v^p(x), \quad p = 1/m. \]

This is more interesting for \( m < 1, \) specially for \( m = (n - 2)/(n + 2). \)
Parabolic to Elliptic

- **Semigroup solution / mild solution.** The typical product of functional discretization schemes: \( u = \{u_n\}_n, u_n = u(\cdot, t_n), \)

\[
    u_t = \Delta \Phi(u), \quad \frac{u_n - u_{n-1}}{h} - \Delta \Phi(u_n) = 0.
\]

Now put \( f := u_{n-1}, u := u_n, \) and \( \nu = \Phi(u), u = \beta(\nu): \)

\[
    -h\Delta \Phi(u) + u = f, \quad -h\Delta \nu + \beta(\nu) = f.
\]

"Nonlinear elliptic equations"; Crandall-Ligget Theorems authors: Benilan, Brezis; Vazquez; Ambrosio, Savarè, Nochetto; Brezis, Ponce, Marcus

- **Separation of variables.** Put \( u(x, t) = F(x)G(t). \) Then PME gives

\[
    F(x)G'(t) = G^m(t)\Delta F^m(x),
\]

so that \( G'(t) = -G^m(t), \) i.e., \( G(t) = (m - 1)t^{-1/(m-1)} \) if \( m > 1 \) and

\[
    -\Delta F^m(x) = F(x),
    -\Delta \nu(x) = \nu^p(x), \quad p = 1/m.
\]

This is more interesting for \( m < 1, \) specially for \( m = (n - 2)/(n + 2). \)
Parabolic to Elliptic

- **Semigroup solution / mild solution.** The typical product of functional discretization schemes: \( u = \{u_n\}_n, \ u_n = u(\cdot, t_n), \)

\[
\begin{align*}
  u_t &= \Delta \Phi(u), \\
  \frac{u_n - u_{n-1}}{h} - \Delta \Phi(u_n) &= 0.
\end{align*}
\]

Now put \( f := u_{n-1}, \ u := u_n, \) and \( v = \Phi(u), \ u = \beta(v): \)

\[
- h\Delta \Phi(u) + u = f, \quad - h\Delta v + \beta(v) = f.
\]

"Nonlinear elliptic equations"; Crandall-Liggett Theorems authors: Benilan, Brezis; Vazquez; Ambrosio, Savarè, Nochetto; Brezis, Ponce, Marcus

- **Separation of variables.** Put \( u(x, t) = F(x)G(t). \) Then PME gives

\[
F(x)G'(t) = G^m(t)\Delta F^m(x),
\]

so that \( G'(t) = -G^m(t), \) i.e., \( G(t) = (m - 1)t^{-1/(m-1)} \) if \( m > 1 \) and \( -\Delta F^m(x) = F(x), \)

\[
-\Delta v(x) = v^p(x), \ p = 1/m.
\]

This is more interesting for \( m < 1, \) specially for \( m = (n - 2)/(n + 2). \)
Functional Analysis Program. Main facts

- Existence of an evolution semigroup.
  \[ u_0 \mapsto S_t(u_0) = u(t) \]

A key issue is the choice of functional space.
- \( X = L^1(\mathbb{R}^n) \) (Brezis, Benilan, Crandall, 1971)
- \( Y = L^1_{loc}(\mathbb{R}^n) \) with growth conditions (Aronson, Caffarelli, Benilan, Crandall, Pierre, 1980’s)
- \( M = \) Locally bounded measures with a growth conditions (Pierre, 1987)

- Positivity. SMP, Nonnegative data produce positive solutions? NO, there are free boundaries.
- “Smoothing effect”: \( L^p \to L^q \) with \( q > p \geq 1, q = \infty \) OK.
- [DeGiorgi-Moser-Nash’s Program]: Then solutions are \( C^\alpha \) for some \( \alpha > 0 \).
- Then are they \( C^\infty \) smooth? NO, they are only Lipschitz cont. under some conditions, (Caffarelli, Friedman, Vazquez, Wolanski,..., they are only Hölder cont. in general. In other things go wrong (things=Functional Analysis). we are still involved in this question !!
- Theory for equation with two signs is still poorly understood.
- \( u_t = \nabla \cdot (c(u)\nabla u), c(u) = |u|^{m-1} \).
- Cf. Stefan Problem (Athanasopoulos, Caffarelli, Sandier).
Functional Analysis Program. Main facts

- **Existence of an evolution semigroup.**

\[ u_0 \mapsto S_t(u_0) = u(t) \]

A key issue is the choice of functional space.

\[ X = L^1(\mathbb{R}^n) \] (Brezis, Benilan, Crandall, 1971)

\[ Y = L^1_{loc}(\mathbb{R}^n) \] with growth conditions (Aronson, Caffarelli, Benilan, Crandall, Pierre, 1980’s)

\[ M = \text{Locally bounded measures with a growth conditions} \] (Pierre, 1987)

- **Positivity.** SMP, Nonnegative data produce positive solutions? NO, there are free boundaries.

- "Smoothing effect": \( L^p \rightarrow L^q \) with \( q > p \geq 1, q = \infty \) OK.

- [DeGiorgi-Moser-Nash’s Program]: Then solutions are \( C^\alpha \) for some \( \alpha > 0 \).

- Then are they \( C^\infty \) smooth? NO, they are only Lipschitz cont. under some conditions, (Caffarelli, Friedman, Vazquez, Wolanski,..., they are only Hölder cont. in general. In other things go wrong (things=Functional Analysis). we are still involved in this question !!

- Theory for equation with two signs is still poorly understood.

\[ u_t = \nabla \cdot (c(u)\nabla u), \ c(u) = |u|^{m-1}. \]

Cf. Stefan Problem (Athanasopoulos, Caffarelli, Salsa).
Functional Analysis Program. Main facts

- **Existence of an evolution semigroup.**
  \[ u_0 \mapsto S_t(u_0) = u(t) \]

A key issue is the choice of functional space.

- \( X = L^1(\mathbb{R}^n) \) (Brezis, Benilan, Crandall, 1971)
- \( Y = L^1_{loc}(\mathbb{R}^n) \) with growth conditions (Aronson, Caffarelli, Benilan, Crandall, Pierre, 1980’s)
- \( M = \) Locally bounded measures with a growth conditions (Pierre, 1987)

- **Positivity.** SMP, Nonnegative data produce positive solutions? NO, there are free boundaries.
- ”**Smoothing effect**”: \( L^p \rightarrow L^q \) with \( q > p \geq 1, q = \infty \) OK.
- [DeGiorgi-Moser-Nash’s Program]: Then solutions are \( C^{\alpha} \) for some \( \alpha > 0 \).
- Then are they \( C^{\infty} \) smooth? NO, they are only Lipschitz cont. under some conditions, (Caffarelli, Friedman, Vazquez, Wolanski,..., they are only Hölder cont in general. In other things go wrong (things=Functional Analysis). we are still involved in this question !!
- Theory for equation with two signs is still poorly understood.
  \[ u_t = \nabla \cdot (c(u)\nabla u), \quad c(u) = |u|^{m-1}. \]
  Cf. Stefan Problem (Athanasopoulos, Caffarelli, Salsa).
Functional Analysis Program. Main facts

- Existence of an evolution semigroup.
  
  \[ u_0 \mapsto S_t(u_0) = u(t) \]

A key issue is the choice of functional space.

- \( X = L^1(\mathbb{R}^n) \) (Brezis, Benilan, Crandall, 1971)
- \( Y = L^1_{loc}(\mathbb{R}^n) \) with growth conditions (Aronson, Caffarelli, Benilan, Crandall, Pierre, 1980’s)
- \( M = \) Locally bounded measures with a growth conditions (Pierre, 1987)

- Positivity. SMP, Nonnegative data produce positive solutions? NO, there are free boundaries.
- ”Smoothing effect”: \( L^p \rightarrow L^q \) with \( q > p \geq 1, \ q = \infty \) OK.
- [DeGiorgi-Moser-Nash’s Program]: Then solutions are \( C^\alpha \) for some \( \alpha > 0 \).
- Then are they \( C^\infty \) smooth? NO, they are only Lipschitz cont. under some conditions, (Caffarelli, Friedman, Vazquez, Wolanski,..., they are only Hölder cont in general. In other things go wrong (things=Functional Analysis). we are still involved in this question !!
- Theory for equation with two signs is still poorly understood.
  
  \[ u_t = \nabla \cdot (c(u)\nabla u), \ c(u) = |u|^{m-1}. \]

Cf. Stefan Problem (Athanasopoulos, Caffarelli, Salsa)
Functional Analysis Program. Main facts

- Existence of an evolution semigroup.

\[ u_0 \mapsto S_t(u_0) = u(t) \]

A key issue is the choice of functional space.

- \( X = L^1(\mathbb{R}^n) \) (Brezis, Benilan, Crandall, 1971)
- \( Y = L^1_{\text{loc}}(\mathbb{R}^n) \) with growth conditions (Aronson, Caffarelli, Benilan, Crandall, Pierre, 1980's)
- \( M = \) Locally bounded measures with a growth conditions (Pierre, 1987)

- Positivity. SMP, Nonnegative data produce positive solutions? NO, there are free boundaries.
- "Smoothing effect": \( L^p \to L^q \) with \( q > p \geq 1, q = \infty \) OK.
- [DeGiorgi-Moser-Nash’s Program]: Then solutions are \( C^\alpha \) for some \( \alpha > 0 \).
- Then are they \( C^\infty \) smooth? NO, they are only Lipschitz cont. under some conditions, (Caffarelli, Friedman, Vazquez, Wolanski,...), they are only Hölder cont in general. In other things go wrong (things=Functional Analysis). we are still involved in this question !!

- Theory for equation with two signs is still poorly understood.

\[ u_t = \nabla \cdot (c(u) \nabla u), \quad c(u) = |u|^{m-1}. \]

Cf. Stefan Problem (Athanasopoulos, Caffarelli, Salsa).
Functional Analysis Program. Main facts

- Existence of an evolution semigroup.
  \[ u_0 \mapsto S_t(u_0) = u(t) \]

A key issue is the choice of functional space.

- \[ X = L^1(\mathbb{R}^n) \] (Brezis, Benilan, Crandall, 1971)
- \[ Y = L^1_{\text{loc}}(\mathbb{R}^n) \] with growth conditions (Aronson, Caffarelli, Benilan, Crandall, Pierre, 1980’s)
- \[ M = \text{Locally bounded measures with a growth conditions} \] (Pierre, 1987)

- Positivity. SMP, Nonnegative data produce positive solutions? NO, there are free boundaries.

- "Smoothing effect": \[ L^p \to L^q \] with \( q > p \geq 1, q = \infty \) OK.

- [DeGiorgi-Moser-Nash’s Program]: Then solutions are \( C^\alpha \) for some \( \alpha > 0 \).

- Then are they \( C^\infty \) smooth? NO, they are only Lipschitz cont. under some conditions, (Caffarelli, Friedman, Vazquez, Wolanski,..., they are only Hölder cont in general. In other things go wrong (things=Functional Analysis). we are still involved in this question!!

- Theory for equation with two signs is still poorly understood.
  \[ u_t = \nabla \cdot (c(u)\nabla u), \quad c(u) = |u|^{m-1}. \]

Cf. Stefan Problem (Athanasopoulos, Caffarelli, Salsa).
Asymptotic behaviour I
Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x) \, dx + \int \int f \, dx \, dt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] Let $B(x, t; M)$ be the Barenblatt with the asymptotic mass $M; u$ converges to $B$ after renormalization

$$t^\alpha |u(x, t) - B(x, t)| \to 0$$

For every $p \geq 1$ we have

$$\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p - 1).$$

Note: $\alpha$ and $\beta = \alpha/n = 1/(2 + n(m - 1))$ are the zooming exponents as in $B(x, t)$.

Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$.
Asymptotic behaviour I
Nonlinear Central Limit Theorem

Choice of domain: \( \mathbb{R}^n \). Choice of data: \( u_0(x) \in L^1(\mathbb{R}^n) \). We can write

\[
  u_t = \Delta(|u|^{m-1}u) + f
\]

Let us put \( f \in L^1_{x,t} \). Let \( M = \int u_0(x) \, dx + \iint f \, dx \, dt \).

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] Let \( B(x, t; M) \) be the Barenblatt with the asymptotic mass \( M \); \( u \) converges to \( B \) after renormalization

\[
  t^\alpha |u(x, t) - B(x, t)| \to 0
\]

For every \( p \geq 1 \) we have

\[
  \|u(t) - B(t)\|_p = o(t^{-\alpha/p'}) , \quad p' = p/(p - 1).
\]

Note: \( \alpha \) and \( \beta = \alpha/n = 1/(2 + n(m - 1)) \) are the zooming exponents as in \( B(x, t) \).

Starting result by FK takes \( u_0 \geq 0 \), compact support and \( f = 0 \).
Asymptotic behaviour I

Nonlinear Central Limit Theorem

Choice of domain: $\mathbb{R}^n$. Choice of data: $u_0(x) \in L^1(\mathbb{R}^n)$. We can write

$$u_t = \Delta(|u|^{m-1}u) + f$$

Let us put $f \in L^1_{x,t}$. Let $M = \int u_0(x) \, dx + \int \int f \, dx dt$.

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] Let $B(x, t; M)$ be the Barenblatt with the asymptotic mass $M$; $u$ converges to $B$ after renormalization

$$t^\alpha |u(x, t) - B(x, t)| \to 0$$

For every $p \geq 1$ we have

$$\|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p - 1).$$

Note: $\alpha$ and $\beta = \alpha/n = 1/(2 + n(m - 1))$ are the zooming exponents as in $B(x, t)$.

Starting result by FK takes $u_0 \geq 0$, compact support and $f = 0$. 
Asymptotic behaviour I
Nonlinear Central Limit Theorem

Choice of domain: \( \mathbb{R}^n \). Choice of data: \( u_0(x) \in L^1(\mathbb{R}^n) \). We can write

\[
  u_t = \Delta(|u|^{m-1}u) + f
\]

Let us put \( f \in L^1_{x,t} \). Let \( M = \int u_0(x) \, dx + \iint f \, dx \, dt \).

Asymptotic Theorem [Kamin and Friedman, 1980; V. 2001] Let \( B(x, t; M) \) be the Barenblatt with the asymptotic mass \( M \); \( u \) converges to \( B \) after renormalization

\[
  t^\alpha |u(x, t) - B(x, t)| \to 0
\]

For every \( p \geq 1 \) we have

\[
  \|u(t) - B(t)\|_p = o(t^{-\alpha/p'}), \quad p' = p/(p - 1).
\]

Note: \( \alpha \) and \( \beta = \alpha/n = 1/(2 + n(m - 1)) \) are the zooming exponents as in \( B(x, t) \).

Starting result by FK takes \( u_0 \geq 0 \), compact support and \( f = 0 \).
Calculations of entropy rates

- We rescale the function as $u(x, t) = r(t)^n \rho(y \ r(t), s)$ where $r(t)$ is the Barenblatt radius at $t + 1$, and “new time” is $s = \log(1 + t)$. Equation becomes

$$\rho_s = \text{div} \left( \rho (\nabla \rho^{m-1} + \frac{c}{2} \nabla y^2) \right).$$

- Then define the entropy

$$E(u)(t) = \int \left( \frac{1}{m} \rho^m + \frac{c}{2} \rho y^2 \right) dy$$

The minimum of entropy is identified as the Barenblatt profile.

- Calculate

$$\frac{dE}{ds} = - \int \rho |\nabla \rho^{m-1} + cy|^2 dy = -D$$

Moreover,

$$\frac{dD}{ds} = -R, \quad R \sim \lambda D.$$ 

We conclude exponential decay of $D$ and $E$ in new time $s$, which is potential in real time $t$. 
References

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME

  About estimates and scaling
  About asymptotic behaviour. (Following Lyapunov and Boltzmann)

On Nonlinear Diffusion

References

References. 1903: Boussinesq, ~1930: Liebenzon. Muskat, ~1950: Zeldovich, Barenblatt, 1958: Oleinik, ...

Classical work after ~1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME


About estimates and scaling


About asymptotic behaviour. (Following Lyapunov and Boltzmann)


On Nonlinear Diffusion

References


Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME


About estimates and scaling


About asymptotic behaviour. (Following Lyapunov and Boltzmann)


On Nonlinear Diffusion

References

References. 1903: Boussinesq, ∼1930: Liebenzon. Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik, ...
Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME


About estimates and scaling


About asymptotic behaviour. (Following Lyapunov and Boltzmann)


On Nonlinear Diffusion

References


Classical work after ~1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, … Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, …

Books. About the PME


About estimates and scaling


About asymptotic behaviour. (Following Lyapunov and Boltzmann)


On Nonlinear Diffusion

References

References. 1903: Boussinesq, ∼1930: Liebenzon. Muskat, ∼1950: Zeldovich, Barenblatt, 1958: Oleinik, ...

Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

Books. About the PME


About estimates and scaling


About asymptotic behaviour. (Following Lyapunov and Boltzmann)


On Nonlinear Diffusion

References


Classical work after ∼1970 by Aronson, Benilan, Brezis, Crandall, Caffarelli, Friedman, Kamin, Kenig, Peletier, JLV, ... Recent by Carrillo, Toscani, MacCann, Markowich, Dolbeault, Lee, Daskalopoulos, ...

*Books. About the PME*


  *About estimates and scaling*


  *About asymptotic behaviour. (Following Lyapunov and Boltzmann)*


*On Nonlinear Diffusion*

Outline

1 Theories of Diffusion
   - Diffusion
   - Heat equation
   - Linear Parabolic Equations
   - Nonlinear equations

2 Degenerate Diffusion
   - Introduction
   - The basics
   - Planning the theory
   - Asymptotic behaviour
   - References

3 Fast Diffusion Equation
FDE Barenblatt profiles

- We have well-known explicit formulas for self-similar Barenblatt profiles with exponents less than one if $1 > m > (n - 2)/n$:

$$B(x, t; M) = t^{-\alpha} F(x/t^\beta), \quad F(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$

The exponents are $\alpha = \frac{n}{2 - n(1-m)}$ and $\beta = \frac{1}{2 - n(1-m)} > 1/2$.

Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n - 2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.
FDE Barenblatt profiles

- We have well-known explicit formulas for self-similar Barenblatt profiles with exponents less than one if $1 > m > (n - 2)/n$:

\[
B(x, t; M) = t^{-\alpha} F(x/t^\beta), \quad F(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}
\]

The exponents are $\alpha = \frac{n}{2 - n(1-m)}$ and $\beta = \frac{1}{2 - n(1-m)} > 1/2$.

Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n - 2)/n$?
- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

Juan Luis Vázquez (Univ. Autónoma de Madrid) Theory of Nonlinear Diffusion March 28 - April 1, 2010 at TU, Berlin
FDE Barenblatt profiles

- We have well-known explicit formulas for self-similar Barenblatt profiles with exponents less than one if $1 > m > (n - 2)/n$:

$$B(x, t; M) = t^{-\alpha} F(x/t^\beta), \quad F(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$

- The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

- Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n - 2)/n$?

- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.
FDE Barenblatt profiles

- We have well-known explicit formulas for self-similar Barenblatt profiles with exponents less than one if $1 > m > (n - 2)/n$:

$$\mathbf{B}(x, t; M) = t^{-\alpha} \mathbf{F}(x/t^\beta), \quad \mathbf{F}(\xi) = \frac{1}{(C + k\xi^2)^{1/(1-m)}}$$

The exponents are $\alpha = \frac{n}{2-n(1-m)}$ and $\beta = \frac{1}{2-n(1-m)} > 1/2$.

Solutions for $m > 1$ with fat tails (polynomial decay; anomalous distributions)

- Big problem: What happens for $m < (n - 2)/n$?

- Main items: existence for very general data, non-existence for very fast diffusion, non-uniqueness for v.f.d., extinction, universal estimates, lack of standard Harnack.

Juan Luis Vázquez (Univ. Autónoma de Madrid)  Theory of Nonlinear Diffusion  March 28 - April 1, 2010 at TU, Berlin
To be Continued,

Thank you

♠ ♠ ♠

Danke, Gracias
To be Continued,

Thank you

♠ ♠ ♠

Danke, Gracias
To be Continued,
Thank you

Danke, Gracias
To be Continued,
Thank you

♠ ♠ ♠

Danke, Gracias