We are interested in the formal parabolic / pseudoparabolic problem in $H^1_0(\Omega)$:

$$f(\partial_u u) - \Delta u - c_\alpha \partial_u u = g, \quad u(0) = u_0.$$  \hfill (1)

Equation of Sobolev: $\partial_u u - \Delta u - c_\alpha \partial_u u = g$; of Barenblatt: $f(\partial_u u) - \Delta u = g$.

1. Assumptions and main result

If $\Omega \subset \mathbb{R}^n$ is a Lipschitz bounded domain, $\partial\Omega = \Gamma, T > 0$ and $Q = [0,T] \times \Omega$.

The main result is:

A solution to (1) is any $f(\partial_u u) - \Delta u - c_\alpha \partial_u u = g$.

The main result is:

Theorem 1. Denote by $\lambda_1$, the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$ and $\lambda_0 = \frac{1}{c_\alpha}$. Then,

1. If $\epsilon > \epsilon_0 > 0$, there exists a unique solution with the Lipschitz principle:

$$\forall \epsilon > \epsilon_0 > 0, \quad \|\nabla(u - \hat{u})(t)\|_{L^2(0,T; \Omega)}^2 + (\epsilon - \epsilon_0) \int_{\Omega_0} \|\nabla(u_0 - \hat{u}(0))\|^2 \leq C(\Omega; \epsilon_0),$$

2. If $0 < \epsilon < \epsilon_0$, there exists a unique solution with the contraction when $\epsilon = \hat{\epsilon}$:

$$\forall \epsilon < \epsilon_0 > 0, \quad \|\nabla(u - \hat{u})(t)\|_{L^2(0,T; \Omega)}^2 \leq \|\nabla(u_0 - \hat{u}(0))\|^2.$$

If moreover, the support of $f_1$, is bounded, there is a unique solution.

3. If $0 < \epsilon < \epsilon_0$, in $H^1_0(0,T; H^1_0(\Omega))$, solutions are not unique in general.

4. If $u = 0$. If $g \in L^2(\Omega)$ and $f \in C^1(\mathbb{R})$, then there exists a unique solution in $H^1(\Omega) \cap L^2(0,T; H^1_0(\Omega))$ to equation

$$f(\partial_u u) - \Delta u = g, \quad u(0) = u_0.$$

2. Existence and uniqueness in the first case

The uniqueness result is given by the test-function $\partial_1(u - \hat{u})$. The existence leads on a method of time-discretization:

Lemma 1. If $h < \frac{1}{c_\alpha}$ and $(g, f) \in H^1_0(\Omega)$, there exists a unique sequence $(u^n)$ in $H^1_0(\Omega)$ with $u^n = u_0$ et $\forall n \in \mathbb{N}$ such that:

$$\int_{\Omega_0} \left[ \frac{1}{h} \left( u^{n+1} - u^n \right) + \nabla u^{n+1} \cdot \nabla v + c_\alpha \frac{1}{h} \nabla u^{n+1} \cdot \nabla v \right] \leq \|g\|^2_{L^2(\Omega)}.$$

Proof. By using the test-function $\frac{1}{h} (u^n - \hat{u})$.

Then (up to a sub-sequence), $u^n \to u$ in $H^1_0(0,T; H^1_0(\Omega))$, $u^n \to u$ in $H^1_0(0,T; H^1_0(\Omega))$, and $u^{n+1} - u^n \to g$ in $H^1_0(\Omega)$.

Proof. By passing to the limits, $\xi(t)$ is a solution in $H^1_0(\Omega)$ to the variational problem

$$\forall t \in [0,T], \quad \int_{\Omega_0} \xi \partial_1 u^n(t) dx = \frac{1}{h} \nabla u^n \nabla v dx = \|g\|^2_{L^2(\Omega)}.$$

It is unique. Then, $u_n \to u_n$ in $H^1_0(\Omega)$.

Therefore, $u_n \to u$, and $\xi(t)$ is a weak measurable function, thus measurable. Then, $\hat{\xi} \to \xi$ in $L^2(0,T; H^1_0(\Omega))$ there and exists a solution.

We consider $\epsilon > 0$, and $\epsilon_0$.

Uniqueness is similar. The existence leads on a method of perturbation:

For any $\delta \in (0, \epsilon]$, denote by $f' : \epsilon \to [f(\epsilon)] \delta \delta_0$ and $\hat{\xi}(t)$ to the solution to Problem:

$$f_1(\hat{\xi}(t)) - \Delta u - c_\alpha \partial_1 u = g + \hat{\xi}(t), \quad u(0) = u_0.$$  \hfill (2)

Lemma 3. $(f_1(\hat{\xi}(t)))$ is a Cauchy sequence in $L^2(\Omega)$.

3. $(\hat{\xi}(t))$ and $(\hat{\xi}(t))$ are bounded in $L^2(\Omega)$.

One concludes with monotonicity arguments.

5. Application to Barenblatt’s equation

A solution to this equation would be any function $u \in H^1(\Omega) \cap L^2(0,T; H^1_0(\Omega))$ such that $u(0) = u_0$ in $H^1_0(\Omega)$ and, for any $v \in H^1_0(\Omega)$,

$$\int_0^T \|\hat{\xi}(t)\|_{L^2(\Omega)}^2 dx = \int_0^T \|f_1(\hat{\xi}(t)) u + \nabla u \nabla v \|_{L^2(\Omega)}^2 dx \leq \|f_1(\hat{\xi}(t)) u + \nabla u \nabla v \|_{L^2(\Omega)}^2.$$

To prove the existence of a solution, one considers a double perturbation: a singular one by a pseudoparabolic operator with the parameter $\epsilon$, and the Yosida-approximation $f_\epsilon$ of $f$.

One gets a priori estimates on the corresponding solution $u_{\epsilon_0}$:

5. $u_{\epsilon_0}$ is a Cauchy sequence in $L^2(\Omega)$ and $L^\infty(0,T; H^1_0(\Omega))$.

The problem reduces to finding a solution in a specific functional space associated with $\lambda_0$, a positive eigenfunction associated to $\lambda_0$.

Then, $\alpha$ is solution to a differential equation with multiple solutions.

6. On a stochastic perturbation

Let us consider $W = \{w_n, F_n, 0 \leq t \leq T\}$ a standard adapted one-dimensional continuos Brownian motion with $w_0 = 0$.

There exists a unique progressively measurable solution $u \in L^2[0,T \times \Theta, H^1_0(\Omega)]$ with $\partial_1 u - \int_0^T \|\hat{\xi}(t)\|_{L^2(\Omega)}^2 dx$ in $L^2[0,T \times \Theta, H^1_0(\Omega)]$ solution to

$$f_1(u) - \Delta u \epsilon \partial_1 u - \int_0^T \|\hat{\xi}(t)\|_{L^2(\Omega)}^2 dx = 0.$$