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Basic notation and facts

- E, F, G Banach (Hilbert) spaces;
- $\mathcal{L}(E,F) := \{T : E \supset \underbrace{\mathcal{D}(T)}_{\text{subspace}} \to F,\ T \text{ linear}\}, \quad \mathcal{L}(E) := \mathcal{L}(E,E),$

  $\mathcal{D}(T)$ domain, $\quad \mathcal{G}(T)$ graph,
  $\ker T$ kernel, $\quad R(T)$ range;
- $T \in \mathcal{L}(E,F)$ closable $:\Longleftrightarrow \overline{\mathcal{G}(T)} = \mathcal{G}(\overline{T})$ with $\overline{T} \in \mathcal{L}(E,F)$,
  $T \in \mathcal{L}(E,F)$ closed $:\Longleftrightarrow \mathcal{G}(T)$ closed;
- $\mathcal{C}(E,F) := \{T \in \mathcal{L}(E,F) :\ T \text{ closed}\}, \quad \mathcal{C}(E) := \mathcal{C}(E,E);$
- $\mathcal{B}(E,F) := \{T \in \mathcal{L}(E,F) :\ T \text{ bounded}, \mathcal{D}(T) = E\}, \quad \mathcal{B}(E) := \mathcal{B}(E,E);$
- $T \in \mathcal{L}(E)$ closable:

$$\begin{aligned} \rho(T) &:= \{\lambda \in \mathbb{C} : T - \lambda \text{ injective}, (T-\lambda)^{-1} \in \mathcal{B}(E)\} && \text{resolvent set,} \\ \sigma(T) &:= \mathbb{C} \setminus \rho(T) && \text{spectrum,} \\ \sigma_{\mathrm{p}}(T) &:= \{\lambda \in \mathbb{C} : T - \lambda \text{ not injective}\} && \text{point spectrum;} \end{aligned}$$

- $T \in \mathcal{C}(E) \quad \Longrightarrow \quad \rho(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ bijective}\}$ (by closed graph theorem)
- $T \in \mathcal{L}(E,F)$ bounded: T closed $\Longleftrightarrow \mathcal{D}(T)$ closed;
- $\rho(T) \neq \emptyset \quad \Longrightarrow \quad T$ closed;
- $T \in \mathcal{L}(E,F)$ closed: T (semi-) Fredholm $:\Longleftrightarrow$

$$\begin{aligned} & n(T) := \dim \ker T < \infty, && R(T) \text{ closed,} \\ \text{(or)} \quad & d(T) := \operatorname{codim} R(T) < \infty, && (\Longrightarrow R(T) \text{ closed});\end{aligned}$$

  then $\operatorname{ind} T := n(T) - d(T)$ index of T,

$$\sigma_{\mathrm{ess}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ not Fredholm}\};$$

- $T \in \mathcal{L}(E,F)$, $\rho(T) \neq \emptyset$, Ω component of $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(T)$:

$$\Omega \cap \rho(T) \neq \emptyset \quad \Longrightarrow \quad \Omega \cap \sigma(T) \text{ discrete;}$$

- $T \in \mathcal{L}(E)$ sectorial $:\Longleftrightarrow$ $(-\infty, 0) \subset \rho(T)$, $\exists\, M \geq 0 :$ $\|(T-\lambda)^{-1}\| \leq \dfrac{M}{|\lambda|}$, $\lambda \in (-\infty, 0)$,

  $\rightsquigarrow$ fractional powers T^γ for $\gamma \in [0,1]$ defined;

For E, F Hilbert spaces:

- $T \in \mathcal{L}(E,F)$, $\mathcal{D}(T)$ dense in E: $T^* \in \mathcal{L}(F,E)$ Hilbert space adjoint.
- $T \in \mathcal{L}(E)$ accretive $:\Longleftrightarrow$ $\operatorname{Re}(Tx, x) \geq 0$, $x \in \mathcal{D}(T)$;

  $T \in \mathcal{L}(E)$ m-accretive $:\Longleftrightarrow$ T accretive, $\rho(T) \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \neq \emptyset$,

  $\rightsquigarrow$ $\|(T-\lambda)^{-1}\| \leq \dfrac{1}{|\operatorname{Re}\lambda|}$, $\operatorname{Re}\lambda < 0$ $\rightsquigarrow$ T sectorial with $M = 1$,

  $T \in \mathcal{L}(E)$ quasi-(m-)accretive $:\Longleftrightarrow$ $T + \alpha$ (m-)accretive for some $\alpha > 0$.

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