Summerschool 2010 – TU Berlin
Infinite Dimensional Operator Matrices
Theory and Applications

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Problem Sheet 1: Operator theoretic backgound

In the following E, F, G are Banach or Hilbert spaces. Prove the following statements.

1. Unbounded operators.

i) $S \in \mathcal{C}(F,G)$, $T \in \mathcal{B}(E,F) \implies ST \in \mathcal{C}(E,G)$;

ii) $S \in \mathcal{C}(F,G)$ boundedly invertible, $T \in \mathcal{C}(E,F) \implies ST \in \mathcal{C}(E,G)$;

iii) $S \in \mathcal{C}(F,G)$, $n(S) < \infty$, $R(S)$ closed, $T \in \mathcal{C}(E,F) \implies ST \in \mathcal{C}(E,G)$;

iv) $S,T \in \mathcal{L}(E,F)$, $S+T$ densely defined $\implies T^* + S^* \subset (T+S)^*$; "=" if $S \in \mathcal{B}(E,F)$;

v) $S \in \mathcal{L}(F,G)$, $T \in \mathcal{L}(E,F)$, S, ST densely defined $\implies T^*S^* \subset (ST)^*$;
"=" if $S \in \mathcal{B}(F,G)$.

2. Relative boundedness. Let $T \in \mathcal{L}(E,F)$, $S \in \mathcal{L}(E,G)$.

i) T closed, S closable, $\mathcal{D}(T) \subset \mathcal{D}(S) \implies S$ T-bounded;

ii) S T-bounded with T-bound $\delta < 1 \implies S$ $(S+T)$-bounded with $(S+T)$-bound $\leq \dfrac{\delta}{1-\delta}$;

iii) S T-bounded $\iff \exists\, a'_S, b'_S \geq 0 : \ \|Sx\|^2 \leq {a'_S}^2\|x\|^2 + {b'_S}^2\|Tx\|^2, \ x \in \mathcal{D}(T) \quad (\sharp)$;
the T-bound of S is the infimum of all b'_S such that $(\sharp)$ holds for some a'_S.

3. Relative compactness and relative bound 0.

i) E, F reflexive, $T, S \in \mathcal{L}(E,F)$, T or S closable, S T-compact $\implies$
S T-bounded with T-bound 0;

ii) $T \in \mathcal{L}(E)$ sectorial, $S \in \mathcal{L}(E,F)$ closable, $\mathcal{D}(T^\gamma) \subset \mathcal{D}(S)$ for some $\gamma \in (0,1) \implies$
S T-bounded with T-bound 0;

iii) E, F Hilbert spaces, $T \in \mathcal{C}(E)$ densely defined, $S \in \mathcal{L}(E,F)$ closable, $\mathcal{D}(|T|^\gamma) \subset \mathcal{D}(S)$ for some $\gamma \in (0,1) \implies S$ T-bounded with T-bound 0;

4. Classical perturbation theorems.

i) $T, S \in \mathcal{L}(E,F)$, S T-bounded with T-bound < 1:

$T+S$ closable (closed) $\iff$ T closable (closed), (then $\mathcal{D}(\overline{T+S}) = \mathcal{D}(\overline{T})$);

$T+S$ boundedly invertible $\Longleftarrow$ T boundedly invertible, $a_S\|T^{-1}\| + b_S < 1$;

ii) H Hilbert space, $T, S \in \mathcal{L}(H)$, $T = T^* \geq \gamma_T$, $S \subset S^*$, S T-bounded with T-bound $< 1 \Longrightarrow$

$$T + S = (T+S)^* \geq \gamma_T - \max\left\{\frac{a_S}{1-b_S}, a_S + b_S|\gamma_T|\right\};$$

what is the analogous statement for quasi-m-accretive T?

iii) $T \in \mathcal{C}(E)$, $S \in \mathcal{L}(E)$, S T-compact $\Longrightarrow$

$$\sigma_{\mathrm{ess}}(T+S) = \sigma_{\mathrm{ess}}(T), \qquad \mathrm{ind}\,(T+S-\lambda) = \mathrm{ind}\,(T-\lambda) \text{ for } \lambda \notin \sigma_{\mathrm{ess}}(T).$$