VolkswagenStiftung

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Infinite Dimensional Operator Matrices
Theory and Applications

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Problem Sheet 2
Examples of differential operators

Below we use the Sobolev spaces $W_2^k(\mathbb{R}^n) := \{f \in L_2(\mathbb{R}^n) : f^{(i)} \in L_2(\mathbb{R}^n), i = 1, 2, .., k\}$, $k \in \mathbb{N}$. Prove the following statements.

5. Useful inequalities.

i) *Hardy inequality* ($n \geq 3$): $\displaystyle\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2}\,\mathrm{d}x \leq \frac{4}{(n-2)^2}\int_{\mathbb{R}^n} |\nabla f(x)|^2\,\mathrm{d}x, \quad f \in C_0^\infty(\mathbb{R}^n).$

ii) *Kato's inequality for accretive operators A.* $\|Ax\|^2 \leq 2\|x\|\|A^2x\|, \quad x \in \mathcal{D}(A^2).$

iii) *Hardy-Littlewood inequality.* $\|f'\|^2_{L_2(\mathbb{R})} \leq \|f\|_{L_2(\mathbb{R})}\|f''\|_{L_2(\mathbb{R})}, \quad f \in W_2^2(\mathbb{R}).$

6. Schrödinger operators in $\mathbb{R}^n$ ($n \geq 3$).

Let $V: \mathbb{R}^n \to \mathbb{R}$ be measurable and $V \in \mathcal{C}\big(L_2(\mathbb{R}^n)\big)$ the corr. maximal multiplication operator.

i) V is $-\Delta$-bounded with Δ-bound 0 if $|V(x)| \leq C\left(\dfrac{1}{|x|} + 1\right)$ for almost all $x \in \mathbb{R}^n$;

ii) V is $-\Delta$-compact if, in addition, $\displaystyle\lim_{x\to\infty} V(x) = 0$;

iii) $-\Delta + V$ is positive if $V(x) \geq -\dfrac{(n-2)^2}{4|x|^2}$ for almost all $x \in \mathbb{R}^n$.

7. Dirac operators.

Define the Pauli spin matrices $\sigma_j \in M_2(\mathbb{C})$ and the Dirac matrices $\alpha_j, \beta \in M_4(\mathbb{C})$ by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3, \quad \beta := \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}.$$

Denote the free Dirac operator in $\big(L_2(\mathbb{R}^3)\big)^4$ by

$$D_0 = \frac{1}{\mathrm{i}}c\alpha\cdot\nabla + \beta\, mc^2 = \frac{1}{\mathrm{i}}\sum_{j=1}^{3} c\alpha_j \frac{\partial}{\partial x_j} + \beta\, mc^2, \quad \mathcal{D}(D_0) := \big(W_2^1(\mathbb{R}^3)\big)^4,$$

and for measurable $v\colon \mathbb{R}^3 \to \mathbb{R}$ by $V = vI \in \mathcal{C}\big((L_2(\mathbb{R}^3))^4\big)$ the maximal multiplication operator.

i) D_0 is self-adjoint with $\sigma(D_0) = \sigma_{\mathrm{ess}}(D_0) = \big(-\infty, -mc^2\big] \cap \big[mc^2, \infty\big)$.

ii) V is D_0-bounded with D_0-bound $\le 2\gamma$ if $|V(x)| \le \dfrac{\gamma}{|x|}$ for almost all $x \in \mathbb{R}^3$;

iii) $V(x) = \dfrac{\gamma}{|x|} I$, $x \in \mathbb{R}^3$ (Coulomb potential), is not D_0-compact and yet $\sigma_{\mathrm{ess}}(D) = \sigma_{\mathrm{ess}}(D_0) = \big(-\infty, -mc^2\big] \cap \big[mc^2, \infty\big)$. (Hint: consider the differences of the resolvents!)

8. Klein-Gordon equation. Let $\mathcal{H}$ be a Hilbert space, $H_0 \ge m^2 > 0$ self-adjoint in $\mathcal{H}$, and V symmetric in $\mathcal{H}$ with $\mathcal{D}(H_0^{1/2}) \subset \mathcal{D}(V)$. The abstract Klein-Gordon equation in $\mathcal{H}$ is given by

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}\,V\right)^2 + H_0\right) u = 0;$$

in the physical case, $H_0 = -\Delta + m^2$ and V is a real-valued potential in $\mathcal{H} = L_2(\mathbb{R}^n)$. Set $S := V H_0^{-1/2}$ and assume that $1 \in \rho(S^*S)$.

i) The Klein-Gordon equation is equivalent to the two different first order systems

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathrm{i}\,\mathcal{A}_i \mathbf{x}, \quad i = 1, 2, \qquad \mathcal{A}_1 = \begin{pmatrix} 0 & I \\ H_0 - V^2 & 2V \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} V & I \\ H_0 & V \end{pmatrix}.$$

ii) $H := H_0^{1/2}(I - S^*S)H_0^{1/2}$ is self-adjoint and boundedly invertible in $\mathcal{H}$.

iii) $\mathcal{A}_1$ with $\mathcal{D}(\mathcal{A}_1) = \mathcal{D}(H) \times \mathcal{D}(H_0^{1/2})$ is closed and boundedly invertible.

iv) $\mathcal{A}_2$ is closable and $\mathcal{A}_1 = W \overline{\mathcal{A}_2} W^{-1}$ with $W = \begin{pmatrix} I & 0 \\ V & I \end{pmatrix}$.

v) $V(x) = \dfrac{\gamma}{|x|}$, $x \in \mathbb{R}^n$ (Coulomb potential) satisfies the assumptions if $|\gamma| < \dfrac{n-2}{2}$.