Problem Sheet 3
Block operator matrices and quadratic numerical range

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Quiz. What is the shape of the numerical range of a 2×2 matrix?
Hint: Try to look at special cases such as

$$\mathrm{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad \mathrm{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}!$$

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In the following let always $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces, $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$, and

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}_1 \times \mathcal{D}_2, \quad \mathcal{D}_1 = \mathcal{D}(A) \cap \mathcal{D}(C), \quad \mathcal{D}_2 = \mathcal{D}(B) \cap \mathcal{D}(D);$$

here all entries $A \in \mathcal{L}(\mathcal{H}_1)$, $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$, $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, $D \in \mathcal{L}(\mathcal{H}_2)$ are closable and densely defined and such that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Prove the following statements.

9. Simple properties of the quadratic numerical range (QNR).

i) $\mathcal{A}$ bounded $\Longrightarrow W^2(\mathcal{A}) \subset K_{\|\mathcal{A}\|}(0) := \{z \in \mathbb{C} : |z| \le \|\mathcal{A}\|\}$; $\dim \mathcal{H} < \infty \Longrightarrow W^2(\mathcal{A})$ closed;

ii) $W^2(\alpha\mathcal{A} + \beta) = \alpha W^2(\mathcal{A}) + \beta$, α, $\beta \in \mathbb{C}$;

iii) $U = \operatorname{diag}(U_1, U_2)$, $U_1 \in \mathcal{L}(\mathcal{H}_1)$, $U_2 \in \mathcal{L}(\mathcal{H}_2)$ unitary $\Longrightarrow W^2(U^{-1}\mathcal{A}U) = W^2(\mathcal{A})$;

iv) $\overline{W(A)} \cap \overline{W(D)} = \emptyset$, $2\sqrt{\|B\|\,\|C\|} < \operatorname{dist}\big(W(A), W(D)\big) \Longrightarrow \overline{W^2(\mathcal{A})}$ has two components.

10. Self-adjoint and $\mathcal{J}$-self-adjoint block operator matrices.

i) $\mathcal{A}$ symmetric $\Longleftrightarrow A|_{\mathcal{D}_1} \subset (A|_{\mathcal{D}_1})^*$, $B|_{\mathcal{D}_2} \subset (C|_{\mathcal{D}_1})^*$, $D|_{\mathcal{D}_2} \subset (D|_{\mathcal{D}_2})^*$;

ii) if one of the following holds:

(dd) A, D Fredholm, C A-compact, B D-compact, B^* A^*-compact, C^* D^*-compact,
(odd) B, C Fredholm, A C-compact, D B-compact, A^* B^*-compact, D^* C^*-compact,

then

$$\mathcal{A}^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix};$$

iii) find conditions for diagonally dominant and off-diagonally dominant block operator matrices to be self-adjoint!

iv) what are the analogues for $\mathcal{J}$-symmetric / $\mathcal{J}$-self-adjoint block operator matrices with

$$\mathcal{J} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}?$$

11. Block operator matrices with real QNR.

i) $\mathcal{A} = \mathcal{A}^* \implies W^2(\mathcal{A}) \subset \mathbb{R}$; $\mathcal{A}$ $\mathcal{J}$-self-adjoint $\implies W^2(\mathcal{A})$ symmetric to $\mathbb{R}$;

ii) B, C closed, $\not\equiv 0$, $(By,x)(Cx,y) \in \mathbb{R}$, $x \in \mathcal{D}(C)$, $y \in \mathcal{D}(B) \implies \exists\, \gamma \in \mathbb{R} : C \subset \gamma B^*$;

iii) if $\dim \mathcal{H}_1$, $\dim \mathcal{H}_2 \geq 2$, B, $C \not\equiv 0$, $\mathcal{D}(B) \cap \mathcal{D}(D)$ core of B, $\mathcal{D}(A) \cap \mathcal{D}(C)$ core of C, then

$$W^2(\mathcal{A}) \subset \mathbb{R} \implies \mathcal{A} \text{ similar to a symmetric or to a } \mathcal{J}\text{-symmetric BOM.}$$

12. Spectral inclusion.

i) $T \in \mathcal{L}(E)$ closable $\implies \sigma_{\mathrm{p}}(\overline{T}) \subset \sigma_{\mathrm{app}}(T)$, $\sigma_{\mathrm{app}}(\overline{T}) = \sigma_{\mathrm{app}}(T)$;

ii) $\mathcal{A}$ off-diagonally dominant of order 0, B, C boundedly invertible $\implies \sigma_{\mathrm{app}}(\mathcal{A}) \subset \overline{W^2(\mathcal{A})}$;

iii) find an example showing that ii) need not hold if one of B, C is not boundedly invertible.