

## Problem Sheet 4

### Schur complements

Let  $E, F, G, H$  denote Banach spaces,  $\mathcal{H}_1, \mathcal{H}_2$  Banach or Hilbert spaces,  $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ , and

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}_1 \times \mathcal{D}_2, \quad \mathcal{D}_1 = \mathcal{D}(A) \cap \mathcal{D}(C), \quad \mathcal{D}_2 = \mathcal{D}(B) \cap \mathcal{D}(D);$$

here all entries  $A \in \mathcal{L}(\mathcal{H}_1)$ ,  $B \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ ,  $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ ,  $D \in \mathcal{L}(\mathcal{H}_2)$  are closable and densely defined and such that  $\mathcal{D}(\mathcal{A})$  is dense in  $\mathcal{H}$ . Define the *Schur complements* of  $\mathcal{A}$

$$\begin{aligned} S_1(\lambda) &:= A - \lambda - B(D - \lambda)^{-1}C, \quad \lambda \in \rho(D), \\ S_2(\lambda) &:= D - \lambda - C(A - \lambda)^{-1}B, \quad \lambda \in \rho(A). \end{aligned}$$

Prove the following statements.

#### 13. $\mathcal{A}$ -invariant subspaces and Riccati equations.

- i) For  $K_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ ,  $K_2 \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$  the following are equivalent:
  - (a)  $I_{\mathcal{H}_2} - K_1 K_2$  boundedly invertible,    (c)  $W := \begin{pmatrix} I_{\mathcal{H}_1} & K_2 \\ K_1 & I_{\mathcal{H}_2} \end{pmatrix}$  boundedly invertible,
  - (b)  $I_{\mathcal{H}_1} - K_2 K_1$  boundedly invertible,    (d)  $\mathcal{H} = \mathcal{G}(K_1) \dot{+} \mathcal{G}^{\text{inv}}(K_2)$ ,
 where  $\mathcal{G}^{\text{inv}}(K_2) = \{(K_2 x \ x)^t : x \in \mathcal{H}_2\}$ ;
- ii) for  $K_1 \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ :  $\mathcal{G}(K_1)$   $\mathcal{A}$ -invariant  $\iff (K_1 A - D K_1 + K_1 B K_1 - C)x = 0$ ,  
 $x \in \mathcal{D}_1 \cap K_1^{-1}(\mathcal{D}_2)$ .

#### 14. Schur complements and spectrum.

- i)  $\Omega \subset \mathbb{C}$  open,  $S : \Omega \rightarrow \mathcal{C}(E, F)$ ,  $T : \Omega \rightarrow \mathcal{C}(G, H)$ : if  $S, T$  are *equivalent*, i.e.  $\exists W_1(\lambda) \in \mathcal{B}(F, H)$ ,  $W_2(\lambda) \in \mathcal{B}(G, E)$  boundedly invertible with  $T(\lambda) = W_1(\lambda)S(\lambda)W_2(\lambda)$ ,  $\lambda \in \Omega$ , then

$$\sigma(T) = \sigma(S), \quad \sigma_p(T) = \sigma_p(S), \quad \sigma_c(T) = \sigma_c(S), \quad \sigma_r(T) = \sigma_r(S);$$

- ii) if  $\mathcal{D}(A) \subset \mathcal{D}(C)$  and  $(A - \lambda)^{-1}B$  is bounded on  $\mathcal{D}(B)$  for all  $\lambda \in \Omega = \rho(A)$ , then  $\lambda \mapsto \mathcal{A} - \lambda$  and  $\lambda \mapsto \begin{pmatrix} A - \lambda & 0 \\ 0 & S_2(\lambda) \end{pmatrix}$  are equivalent on  $\Omega$ ;

- iii)  $\mathcal{A}$  diagonally dominant  $\implies \rho(S_2) \cap \rho(D)$  open.

## 15. Essential spectrum.

- i)  $T \in \mathcal{L}(E, F)$  densely defined, Fredholm,  $S \in \mathcal{L}(G, E)$  Fredholm  $\implies TS$  Fredholm,  $\text{ind } TS = \text{ind } T + \text{ind } S$ ;
- ii)  $T, \tilde{T} \in \mathcal{C}(E)$ ,  $\exists \lambda \in \rho(T) \cap \rho(\tilde{T})$ :  $(T - \lambda)^{-1} - (\tilde{T} - \lambda)^{-1}$  compact  $\implies \sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(\tilde{T})$ ;
- iii)  $A = A^*$ ,  $D = D^*$ ,  $C = B^*$ ,  $\mathcal{D}(B) \cap \mathcal{D}(D)$  core of  $D$ ,  $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(B^*)$ ,  $\exists \mu \in \mathbb{C} \setminus \mathbb{R}$ :  $(D - \mu)^{-1}B^*(A - \mu)^{-1}$  compact  $\implies \sigma_{\text{ess}}(\overline{\mathcal{A}}) = \sigma_{\text{ess}}(A) \cup \sigma_{\text{ess}}(D - B^*(A - \mu)^{-1}B)$ .

## 16. Schur complements and QNR.

- i)  $\mathcal{D}(A) \subset \mathcal{D}(C)$ ,  $g \in \mathcal{D}(B) \cap \mathcal{D}(D)$ ,  $g \neq 0$ ,  $Bg \neq 0$ ,  $\lambda \in \rho(A)$   $\implies$   

$$\Delta((A - \lambda)^{-1}Bg, g; \lambda) = (Bg, (A - \lambda)^{-1}Bg)(S_2(\lambda)g, g).$$

Let now  $\mathcal{A}$  be diagonally dominant,  $A = A^*$ ,  $D = D^*$ .

- ii)  $W^2(\mathcal{A}) \subset \mathbb{R}$ ,  $W^2(\mathcal{A})$  has two connected components  $\Lambda_{\pm}(\mathcal{A})$ ,  $\Lambda_{-}(\mathcal{A}) < \gamma < \Lambda_{+}(\mathcal{A})$ ,  $\max \sigma(D) < \gamma < \min \sigma(A)$   $\implies S_2(\gamma) < 0$ ;
- iii)  $C \subset \pm B^*$ ,  $\gamma \in \rho(A) \cap \mathbb{R}$   $\implies S_2(\gamma)$  symmetric;  
find situations where  $S_2(\gamma)$  is self-adjoint;
- iv) formulate the analogues for the first Schur complement  $S_1$ .